Solution Set 3

1.

(a)

$$F(Q) = \frac{\tilde{\rho}(Q)}{q}, \quad q=1 \text{ here} \Rightarrow F(Q) = \tilde{\rho}(Q)$$

$$\tilde{\rho}(Q) = \int d^3x \, e^{-i\vec{x}\cdot\vec{Q}} \rho(r)$$

$$= 2\pi \int_0^\infty r^2 dr \, \rho(r) \int_{-1}^{1} d\cos\theta \, e^{-irQ\cos\theta}$$

$$= 2\pi \int_0^\infty r^2 dr \, \rho(r) \frac{i}{rQ} \left(e^{-irQ} - e^{irQ} \right)$$

$$= -\frac{2\pi i}{Q} \int_0^\infty r\, dr \, \rho(r) \left(e^{irQ} - e^{-irQ} \right) \quad //$$

$$= \frac{4\pi}{Q} \int_0^\infty r\, dr\, \rho(r) \sin(Qr)$$

(b)

$$\rho(r) = \frac{1}{(2\pi a^2)^{3/2}} e^{-r^2/2a^2}$$

$$\tilde{\rho}(Q) = -\frac{2\pi i}{2Q} \frac{1}{(2\pi a^2)^{3/2}} \int_{-\infty}^{\infty} r\, dr\, e^{-r^2/2a^2} \left(e^{irQ} - e^{-irQ} \right)$$

$$= -\frac{\pi i}{Q} \frac{1}{(2\pi a^2)^{3/2}} (-i) \frac{d}{dQ} \int_{-\infty}^{\infty} dr\, e^{-\frac{r^2}{2a^2}} \left(e^{irQ} + e^{-irQ} \right)$$

$$= -\frac{\pi}{Q} \frac{1}{(2\pi a^2)^{3/2}} \frac{d}{dQ} \left\{ \int_{-\infty}^{\infty} dr\, e^{\frac{-r^2}{2a^2} + irQ} + \int_0^\infty dr\, e^{-\frac{r^2}{2a^2} - irQ} \right\}$$

$$\Big\Vert \quad \int_{-\infty}^{\infty} dr \; e^{-\frac{r^2}{2a^2}+br} = ?$$

$$-\frac{r^2}{2a^2}+br = \frac{1}{2a^2}\left(-r^2+2a^2br\right)$$

$$= \frac{1}{2a^2}\left(-r^2+2a^2br - a^4b^2\right) + \frac{a^2b^2}{2}$$

$$= -\frac{1}{2a^2}\left(r-a^2b\right)^2 + \frac{a^2b^2}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} dr \; e^{-\frac{r^2}{2a^2}+br} = e^{\frac{a^2b^2}{2}} \int_{-\infty}^{\infty} dr \; e^{-\frac{1}{2a^2}(r-a^2b)^2}$$

$$= a\sqrt{2\pi}\; e^{\frac{a^2b^2}{2}} \quad \Big\Vert$$

$$\tilde{p}(Q) = -\frac{\pi}{Q}\frac{1}{(2\pi a^2)^{3/2}}\frac{d}{dQ}\left\{ a\sqrt{2\pi}\; e^{-a^2Q^2/2} + a\sqrt{2\pi}\; e^{-a^2Q^2/2}\right\}$$

$$= -\frac{2\pi}{Q}\frac{1}{(2\pi a^2)^{3/2}}\, a\sqrt{2\pi}\; \frac{d}{dQ}\, e^{-a^2Q^2/2}$$

$$= -\frac{2\pi}{Q}\frac{1}{(2\pi a^2)^{3/2}}\, a\sqrt{2\pi}\; (-a^2Q)\, e^{-a^2Q^2/2}$$

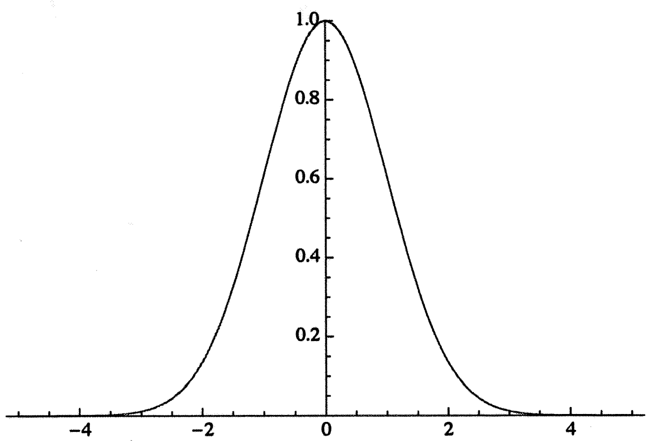
$$= e^{-a^2Q^2/2}$$

$$\boxed{F(Q) = e^{-a^2Q^2/2}}$$

1. c

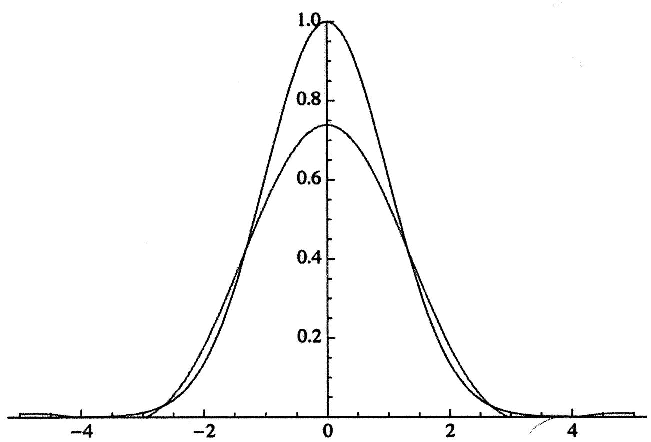
```
In[640]:= Rho1c[r_, a_] := 1 / ((2 Pi a^2) ^ (3 / 2)) Exp[-r^2 / (2 a^2)];
          NumFormFactor[q_, a_] := 4 Pi / q * NIntegrate[r * Rho1c[r, a] * Sin[r * q], {r, 0, Rupper}];
          ExactFormFactor[q_, a_] := Exp[-a^2 q^2 / 2];

In[643]:= Rupper = 4;
          a = 1;
          Plot[{ExactFormFactor[q, a], NumFormFactor[q, a]}, {q, -5, 5}, PlotRange -> {0, 1}]
```

Out[645]=




```
In[646]:= Rupper = 2;
          a = 1;
          Plot[{ExactFormFactor[q, a], NumFormFactor[q, a]}, {q, -5, 5}, PlotRange -> {0, 1}]
```

Out[648]=



(d)

$$\rho(r) = A \frac{1}{1+e^{b(r-a)}}$$

$$\int d^3r\, \rho(r) = 1$$

$$\Rightarrow 1 = 4\pi A \int_0^\infty dr \frac{1}{1+e^{b(r-a)}}$$

$$z = e^{br}$$

$$\Rightarrow \frac{dz}{dr} = b e^{br} = bz$$

$$\Rightarrow \quad dr = \frac{1}{bz} dz$$

$$\frac{1}{4\pi} = A \int_1^\infty \frac{dz}{bz} \frac{1}{1+e^{-ab}z}$$

$$= \frac{A}{b} \log\left(\frac{z}{1+e^{-ab}z}\right)\Bigg|_1^\infty$$

$$= \frac{A}{b}\left[\log\left(\frac{1}{e^{-ab}}\right) - \log\left(\frac{1}{1+e^{-ab}}\right)\right]$$

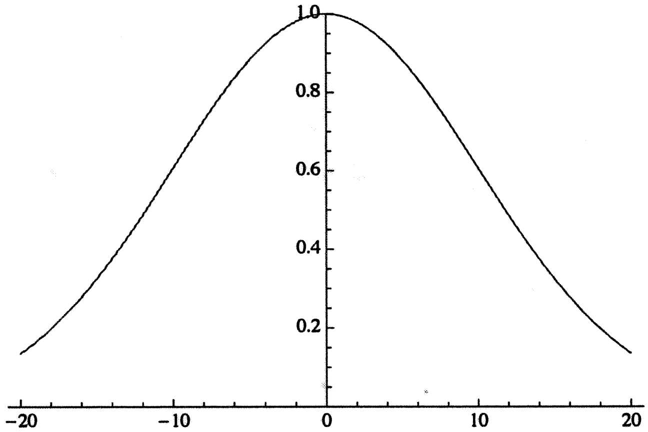
$$= \frac{A}{b}\left[\log\left(\frac{1+e^{-ab}}{e^{-ab}}\right)\right]$$

$$A = \frac{b}{4\pi} \frac{1}{\log\left(\frac{1+e^{-ab}}{e^{-ab}}\right)}$$

-5-

```
In[649]:= Rupper = 1;
          a = 0.1;
          Plot[{ExactFormFactor[q, a], NumFormFactor[q, a]}, {q, -20, 20}, PlotRange → {0, 1}]
```

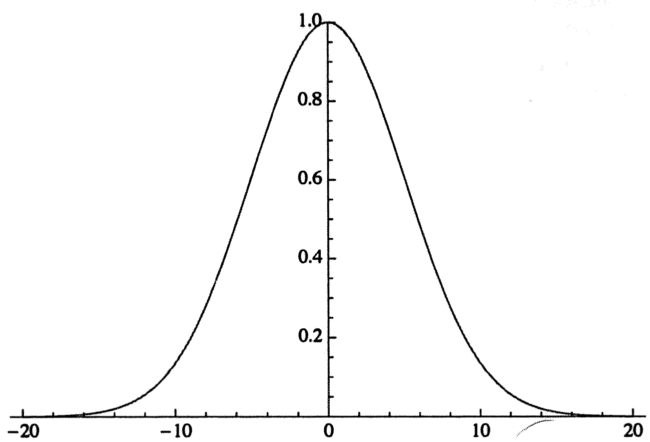
Out[651]=



1. d

```
In[362]:= Rhold[r_, b_] := norm[b] / (1 + Exp[b (r - 1)]);
          norm[b_] := b / (4 Pi) * 1 / (Log[(1 + Exp[-b]) / Exp[-b]]);
          NumFormFactor2[q_, b_] := 4 Pi / q * NIntegrate[r * Rholc[r, b] * Sin[r * q], {r, 0, Rupper}];

In[368]:= b = 0.2;
          Rupper = 10;
          Plot[NumFormFactor2[q, b], {q, -20, 20}, PlotRange → {0, 1}]
```

Out[370]=

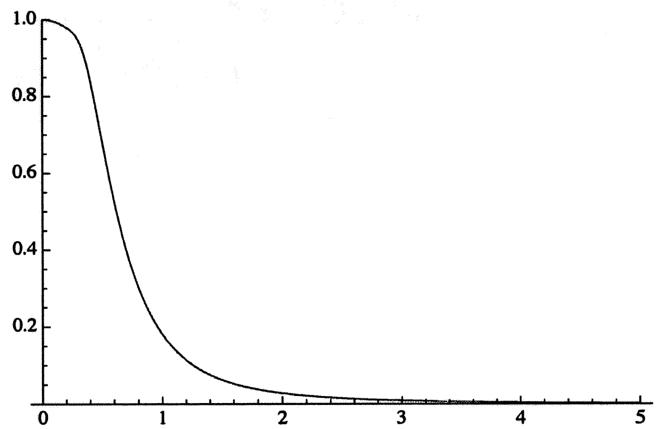


This form factor has no 0 as a function of Q.

In[652]:=
```
Clear[b];
q = 1;
Plot[NumFormFactor2[q, b], {b, 0, 5}, PlotRange → {0, 1}]
```

NIntegrate::ncvb :
NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in r near {r} = {0.666031}. NIntegrate obtained 0.` and 0.` for the integral and error estimates. ≫

Out[654]=



2. The radial Schrodinger equation is

(a) $$r^2 \frac{d^2}{dr^2} f_k(r) + 2r \frac{d}{dr} f_k(r) + \left(r^2 k^2 - 2mr^2 V(r)\right) f_k(r) - l(l+1) f_k(r) = 0 \qquad (1)$$

|| Let $g_k(r) = r f_k(r) \Rightarrow f_k(r) = \frac{1}{r} g_k(r)$,

$$\frac{d}{dr} f_k = \frac{d}{dr}\left(\frac{1}{r} g_k\right) = -\frac{1}{r^2} g_k + \frac{1}{r} g_k'$$

$$\frac{d^2}{dr^2} f_k = \frac{d}{dr}\left(-\frac{1}{r^2} g_k + \frac{1}{r} g_k'\right)$$

$$= \frac{2}{r^3} g_k - \frac{2}{r^2} g_k' + \frac{1}{r} g_k'' \quad ||$$

$$\Rightarrow \cancel{\frac{2}{r} g_k} - \cancel{2 g_k'} + r g_k'' - \cancel{\frac{2}{r} g_k} + \cancel{2 g_k'} + \left(r^2 k^2 - 2mr^2 V(r)\right) \frac{1}{r} g_k - \frac{l(l+1)}{r} g_k = 0$$

$$g_k''(r) + \left(r^2 k^2 - 2mr^2 V(r)\right) \frac{1}{r^2} g_k - \frac{l(l+1)}{r^2} g_k = 0 \qquad (2)$$

<u>Boundary conditions</u>,

The radial wave function $f_k(r)$ must be finite at $r = 0$

$\Rightarrow \frac{1}{r} g_k(r) \underset{r \to 0}{=} \text{finite} \Rightarrow g_k(r) = 0$ at $r = 0$.

The continuity of f_k and f_k' at $r = a$ imply continuity of $g_k(r)\big|_{r=a}$ since $f_k(a) = \frac{1}{a} g_k(a) = \text{const} \times g_k(a)$, and

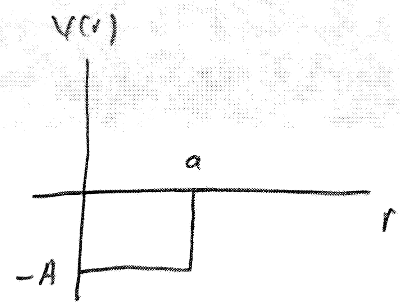
$$f_k'(a) = -\frac{1}{a^2} g_k(a) + \frac{1}{a} g_k'(a).$$

This is solved in Griffiths.

(b) $l=0$, bound states. $\Rightarrow$ $E<0$, $k=\sqrt{-2mE}>0$

$$(2) \rightarrow \quad g_k''(r) + \left[2mE - 2mV(r)\right] g_k = 0 \qquad (3)$$

$$V(r) = \begin{cases} -A, & r<a \\ 0, & r>a \end{cases}$$



For $r<a$

$$\frac{d^2 g_k}{dr^2} + \left(-k^2 + 2mA\right) g_k = 0$$

$$g_k(r) = C\sin(\tilde{k}r) + D\cos(\tilde{k}r), \quad \tilde{k} = \sqrt{-k^2+2mA} \qquad (4)$$

$$g_k(0) = 0 \Rightarrow D = 0 \Rightarrow g_k(r) = C\sin(\tilde{k}r), \quad r<a$$

For $r>a$

$$\frac{d^2 g_k}{dr^2} = -2mE\, g_k = k^2 g_k$$

$$\Rightarrow g_k(r) = Fe^{-kr} \qquad (5)$$

Continuity of g_k and g_k' at $r=a$ gives

$$g_k(r)\big|_{a^-} = g_k(r)\big|_{a^+} \qquad (6)$$

$$g_k'(r)\big|_{a^-} = g_k'(r)\big|_{a^+} \qquad (7)$$

$$(7)/(6) \Rightarrow \frac{1}{g_k}\frac{dg_k}{dr}\bigg|_{a^-} = \frac{1}{g_k}\frac{dg_k}{dr}\bigg|_{a^+} \qquad (8)$$

$$\Rightarrow \quad \frac{\tilde{k}\cos\tilde{k}a}{\sin\tilde{k}a} = \frac{-k e^{-ka}}{e^{-ka}}$$

$$\Rightarrow \quad -\frac{\tilde{k}}{k} = \tan(\tilde{k}a) \qquad (9)$$

Let $\tilde{k}a = z$, $z_0 = a\sqrt{2mA}$,

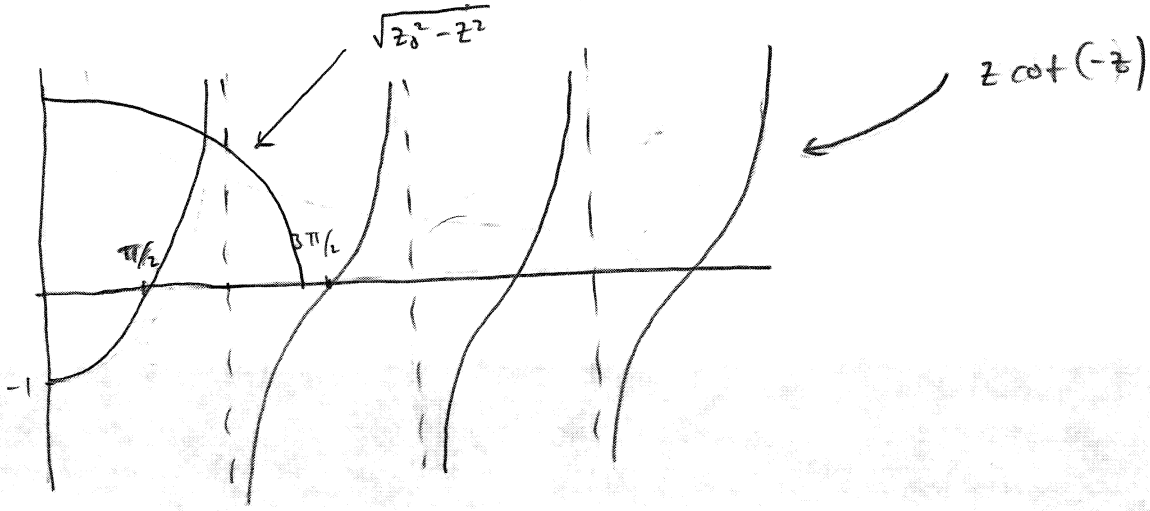
since $\tilde{k}^2 = -k^2 + 2mA$,

$$z^2 = -k^2a^2 + z_0^2$$

$$ka = \sqrt{z_0^2 - z^2}$$

$$(9) \rightarrow \tan(z) = \frac{-z}{\sqrt{z_0^2 - z^2}}$$

$$z\cot(-z) = \sqrt{(z_0^2 - z^2}$$



$$z_0^2 - z^2 = 0 \quad \text{at} \quad z_0 = z^* \Rightarrow a\sqrt{2mA} = z^*$$

larger $A \Rightarrow$ larger $z^* \Rightarrow$ many points of intersection between $z\cot(-z)$ and $\sqrt{z_0^2 - z^2}$.

There will be no bound state for $z^* < \frac{\pi}{2}$

$\Rightarrow a\sqrt{2mA} \leq \frac{\pi}{2}$

A new bound state is added when $z_0^2 - z^2$ hits a new zero of $z\cot(-z)$ $\Rightarrow$ $z^* = a\sqrt{2mA} = \frac{\pi}{2} + n\pi$ $\Rightarrow$ $\boxed{\sqrt{2mA} = \frac{\pi}{2a}(2n+1)}$

(c) We solve schrod. eqn. again, this time for $E > 0$

$$g_k''(r) + (k^2 - 2mV)\, g_k = 0$$

- For $r > a$, $V(r) = 0$

$$\Rightarrow \frac{d^2 g_k}{dr^2} = -k^2 g_k$$

$$\Rightarrow g_k(r) = C\sin(kr + \delta_0) \qquad (1)$$

- For $r < a$, $V(r) = -A$

$$\frac{d^2 g_k}{dr^2} + (k^2 + 2mA)\, g_k = 0$$

$$\Rightarrow g_k(r) = D\sin(k'r) + F\cos(k'r), \quad k' = \sqrt{k^2 + 2mA}$$

$$g_k(0) = 0 \Rightarrow F = 0 \Rightarrow g_k(r) = D\sin(k'r) \qquad (2)$$

continuity of g_k & g_k'

$$\Rightarrow \quad \frac{1}{g_k(a^-)} \frac{d g_k(a^-)}{dr} = \frac{1}{g_k(a^+)} \frac{d g_k(a^+)}{dr}$$

$$\Rightarrow \quad \frac{k' \cos(k'a)}{\sin(k'a)} = \frac{k \cos(ka + \delta_0)}{\sin(ka + \delta_0)}$$

$$\Rightarrow \quad k' \tan(ka + \delta_0) = k \tan(k'a)$$

$$\tan(ka + \delta_0) = \frac{k}{k'} \tan(k'a) \qquad (3)$$

$$n\pi + \delta_0(k) = \arctan\left[\frac{k}{k'} \tan(k'a)\right] - ka$$

$$k' = k\sqrt{1 + \frac{2mA}{k^2}}$$

(d) as $k \to \infty$, $k' = k\sqrt{1 + \frac{2mA}{k^2}} \to k\left(1 + \frac{mA}{k^2}\right)$

$$\frac{k}{k'} \to \frac{1}{1 + \frac{mA}{k^2}} = 1 + O\left(\frac{1}{k^2}\right)$$

$$n\pi + \delta_0 \to k'a - ka = (k' - k)a \to \frac{mA}{k} a \to 0 \text{ as } k \to \infty$$

So, $\delta_0(A) = \dfrac{maA}{k} + n\pi$ for large k $\quad \delta_0\Big|_{k \to 0} = 0$

$\delta_0(A) = \Rightarrow \; n\pi = 0 \Rightarrow n = 0 \quad \therefore \; \delta_0(A) = \dfrac{maA}{k}$

In this limit, we have

$$\tan(ka+\delta_0) = \frac{k}{k'}\tan(k'a)$$

$$\Rightarrow \cot(ka+\delta_0) = \frac{k'a}{ka}\cot(k'a)$$

$$ka\cot(ka+\delta_0) = k'a\cot k'a \qquad (4)$$

Let $k'a = z$ (analogous to the bound state case)

$$z_0 = a\sqrt{2mA}$$

$$\Rightarrow z^2 = k'^2a^2 = a^2(k^2+2mA)$$

$$\Rightarrow a^2k^2 = z^2 - 2mAa^2 = z^2 - z_0^2$$

$$(4) \rightarrow \sqrt{z^2-z_0^2}\cot\left(\sqrt{z^2-z_0^2}+\delta_0\right) = z\cot(z)$$

$$\Rightarrow z\cot(-z) = \sqrt{z^2-z_0^2}\cot\left(-\sqrt{z^2-z_0^2}-\delta_0\right) \qquad (5)$$

At large z, i.e. large k, $z_0/z \to 0$

$$\Rightarrow (5) \rightarrow z\cot(-z) = z\cot(-z+\delta_0).$$

But as we showed in (c), at large k $\delta_0 \to 0$, so the left & right hand sides of (5) line up.

At small k, $z \to z_0$, LHS of (5) $\to$ $z_0 \cot(-z_0)$,

$\sqrt{z^2 - z_0^2} = ka \to 0$

(5) $\Rightarrow$ $z_0 \cot(-z_0) = ka \cot(-ka - \delta_0)$

when $A \to 0$, $z_0 = \sqrt{2mA}\, a \to 0$

$\Rightarrow z_0 \cot(-z_0) = -1$ since $x \cot x \sim x\left(\frac{1}{x} - \frac{x}{3} + \cdots\right.$

$\Rightarrow -1 = ka \cot(-ka + \delta_0)$, $ka \to 0$

$= ka \cot(-\delta_0)$

$\Rightarrow -1 = ka \cot \delta_0$

$\Rightarrow \delta_0(k) = ka$ so that $ka \cot ka \big|_{k=0} = 1$

$\Rightarrow \delta_0(0) = 0$

When $z_0 = \frac{\pi}{2} + n\pi$, then $\cot(-z_0) = 0$

$\Rightarrow 0 = ka \cot(-ka - \delta_0)$

$0 = \cot(ka + \delta_0)$, at $k = 0$ we get

$0 = \cot \delta_0$

$\Rightarrow \delta_0 = \frac{\pi}{2} + n\pi$, but $z_0 = \sqrt{2mA}\, a$

so the condition is that $\sqrt{2mA}\, a = \frac{\pi}{2} + n\pi$. This is precisely the condition for bound states. As we increase A, δ_0 shifts from 0 to $\frac{\pi}{2}(2n+1)$.