Solution Set # 2

1.

(a) $$I = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} \frac{g^2}{p^2+a^2}$$, pick z-axis to align with $\vec{x}$

$$= \frac{1}{(2\pi)^3}\int p^2\,dp\, d^2\Omega\; e^{ipx\cos\theta} \frac{g^2}{p^2+a^2}$$

$$= \frac{1}{(2\pi)^2}\int p^2\,dp\, \frac{g^2}{p^2+a^2} \int_{-1}^{1} d(\cos\theta)\; e^{ipx\cos\theta}$$

$$= \frac{1}{(2\pi)^2}\int p^2\,dp\, \frac{g^2}{p^2+a^2}\, \frac{1}{ipx}\, e^{ipx\cos\theta}\Big|_{\cos\theta=-1}^{1}$$

$$= \frac{1}{(2\pi)^2}\int_0^\infty p^2\,dp\, \frac{g^2}{p^2+a^2}\, \frac{1}{ipx}\left(e^{ipx}-e^{-ipx}\right)$$

$$= \frac{g^2}{(2\pi)^2}\,\frac{1}{ix}\int_0^\infty dp\, \frac{p}{p^2+a^2}\left(e^{ipx}-e^{-ipx}\right)$$

|| The integrand, $f(p) = \dfrac{p}{p^2+a^2}\left(e^{ipx}-e^{-ipx}\right)$ is even.

under $p \to -p$ $\quad f(-p) = \dfrac{-p}{p^2+a^2}\left(e^{-ipx}-e^{ipx}\right) = \dfrac{p}{p^2+a^2}\left(e^{ipx}-e^{-ipx}\right)$ ||

$$= \frac{g^2}{(2\pi)^2}\,\frac{1}{2ix}\int_{-\infty}^\infty dp\, \frac{p}{p^2+a^2}\left(e^{ipx}-e^{-ipx}\right)$$

$$I = \frac{g^2}{(2\pi)^2}\,\frac{1}{2ix}\int_{-\infty}^\infty dp\, \frac{p}{(p+ia)(p-ia)}\left(e^{ipx}-e^{-ipx}\right)$$

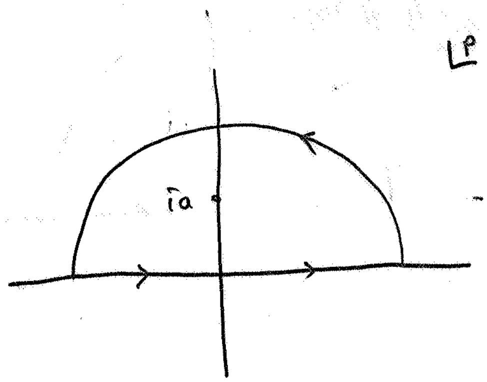
$$= \frac{g^2}{(2\pi)^2}\frac{1}{2ix}\left\{\int_{-\infty}^{\infty} dp\,\frac{p\,e^{ipx}}{(p+ia)(p-ia)} - \int_{-\infty}^{\infty} dp\,\frac{p\,e^{-ipx}}{(p+ia)(p-ia)}\right\}$$

Let $p' = -p$ in the second integral, then

$$-\int_{-\infty}^{\infty} dp\,\frac{p\,e^{-ipx}}{(p+ia)(p-ia)} = -\int_{\infty}^{-\infty}(-dp')\frac{(-p')e^{ip'x}}{(-p'+ia)(-p'-ia)}$$

$$= +\int_{-\infty}^{\infty} dp'\,\frac{p'\,e^{ip'x}}{(p'+ia)(p'-ia)}$$

$$\Rightarrow I = \frac{g^2}{(2\pi)^2}\frac{1}{ix}\int_{-\infty}^{\infty} dp\,\frac{p\,e^{ipx}}{(p+ia)(p-ia)}$$



We can close the contour above and pick up a residue at the pole at $p = ia$.

$$I = \frac{g^2}{(2\pi)^{\cancel{2}}}\frac{1}{\cancel{i}x}\frac{\cancel{ia}\,e^{-ax}}{2\cancel{ia}}(2\pi\cancel{i}) = \frac{g^2}{4\pi}\frac{e^{-ax}}{x}$$

$$\boxed{I = \frac{g^2}{4\pi}\frac{e^{-ax}}{x}}$$

(b) $$I = \int \frac{d^3p}{(2\pi)^3} \, e^{i\vec{p}\cdot\vec{x}} \, \frac{g^2}{p^2 - k^2 - i\epsilon}$$

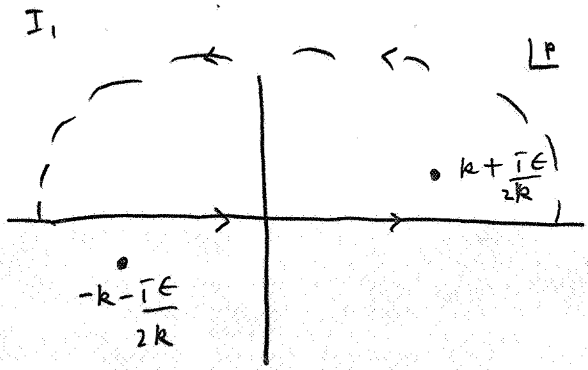
$$I = \frac{g^2}{(2\pi)^2} \frac{1}{2ix} \int_{-\infty}^{\infty} dp \, \frac{p}{p^2 - k^2 - i\epsilon} \left(e^{ipx} - e^{-ipx}\right)$$

after doing the integral over $\cos\theta$, and extending the range of p to $(-\infty, \infty)$. just like in (a).

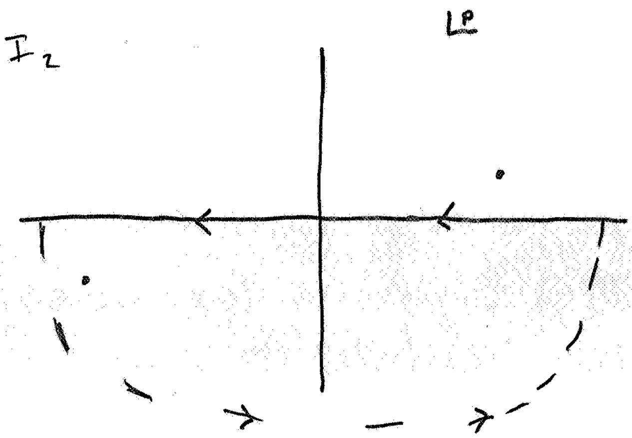
$$p^2 - k^2 - i\epsilon = \left(p - k - \frac{i\epsilon}{2k}\right)\left(p + k + \frac{i\epsilon}{2k}\right) + O(\epsilon^2),$$

$$I = \frac{g^2}{(2\pi)^2} \frac{1}{2ix} \left\{ \int_{-\infty}^{\infty} dp \, \frac{p\, e^{ipx}}{\left(p - k - \frac{i\epsilon}{2k}\right)\left(p + k + \frac{i\epsilon}{2k}\right)} - \int_{-\infty}^{\infty} dp \, \frac{p\, e^{-ipx}}{\left(p - k - \frac{i\epsilon}{2k}\right)\left(p + k + \frac{i\epsilon}{2k}\right)} \right\}$$

I_1



I_2



$$I_1 = 2\pi i \, \frac{\left(k + \frac{i\epsilon}{2k}\right) e^{i\left(k + \frac{i\epsilon}{2k}\right)x}}{\left(-k + \frac{i\epsilon}{2k} + k + \frac{i\epsilon}{2k}\right)}$$

for ϵ very small, we'll keep only 0^{th} term in expansion in ϵ

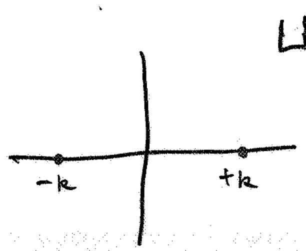
$$I_1 = \frac{2\pi i k\, e^{ikx}}{2k} = \pi i\, e^{ikx}.$$

Similarly, $$I_2 = \frac{2\pi i\,(-k)\, e^{ikx}}{-2k} = \pi i\, e^{ikx}$$

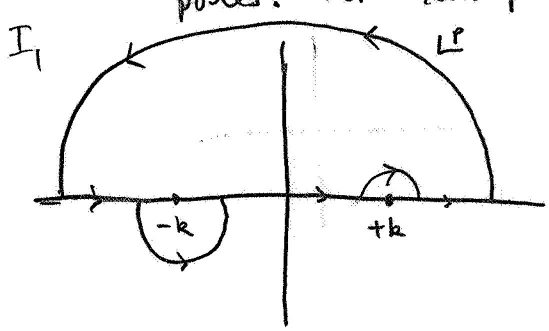
$$\therefore\ I = \frac{g^2}{(2\pi)^2}\,\frac{1}{2ix}\,2\pi i\, e^{ikx}$$

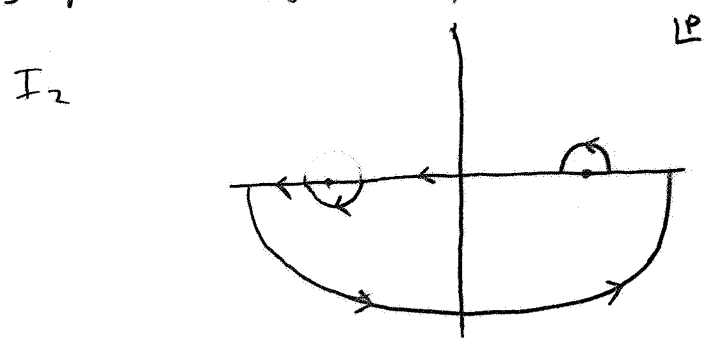
$$\boxed{I = \frac{g^2}{4\pi}\,\frac{e^{ikx}}{x}}$$

Note: The $-i\epsilon$ term plays an important role in enforcing causality. I, which is the Green's function, is a response to a delta function potential, and therefore should always be an outgoing wave. Without the $-i\epsilon$, our contour would overlap with the poles on the real axis:



We have various options as to how to skirt around the poles. For example, let's pick the following contour,





This choice of contours would give

$$I = \frac{g^2}{4\pi} \frac{e^{-ikx}}{x}$$ which is an incoming wave, not

allowed by causality.

Adding the $i\epsilon$ shifts the poles in the correct direction so that there will be no ambiguity about how to skirt around the poles.

(c) $$I = \int \frac{d^d p}{(2\pi)^d} e^{i\vec{p}\cdot\vec{x}} \frac{g^2}{p^2+a^2}$$

$$= \frac{1}{(2\pi)^d} \int_0^\infty dp\, p^{d-1} \int d\theta \sin\theta \sin^{d-3}\theta\, e^{ipx\cos\theta} \frac{g^2}{p^2+a^2} \underbrace{\int d^{d-2}\Omega}_{\mathrm{Vol}(S^{d-2}) = A_{d-1}}$$

$$= \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)} \frac{1}{(2\pi)^d} \int_0^\infty dp\, p^{d-1} \frac{g^2}{p^2+a^2} \int_{-1}^{1} d(\cos\theta) \sin^{d-3}\theta\, e^{ipx\cos\theta}$$

Let $E_1 = \int_{-1}^{1} d(\cos\theta) \sin^{d-3}\theta\, e^{ipx\cos\theta}$,

let $t = \cos\theta \Rightarrow \sin\theta = \sqrt{1-t^2}$,

$$E_1 = \int_{-1}^{1} dt\, (1-t^2)^{\frac{d-3}{2}} e^{ipxt}$$

Eqn. 10.9.4. of NIST :

$$J_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos(zt)\,dt$$

since the integrand is even under $t \to -t$,

$$2\int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos(zt)\,dt = \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} \cos(zt)\,dt$$

$$= \frac{1}{2}\int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} \left(e^{izt}+e^{-izt}\right)$$

$$= \frac{1}{2}\int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} e^{izt}\,dt + \frac{1}{2}\int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} e^{-izt}\,dt$$

$$= \frac{1}{2}\int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} e^{izt}\,dt + \frac{1}{2}\int_{-1}^1 (1-\tau^2)^{\nu-\frac{1}{2}} e^{iz\tau}\,d\tau \quad , \quad \tau = -t$$

$$= \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} e^{izt}\,dt$$

$$\therefore \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} e^{izt}\,dt$$

$$\Rightarrow E_1 = \int_{-1}^1 dt\,(1-t^2)^{\frac{d-2}{2}-\frac{1}{2}} e^{ipxt}$$

$$= \frac{\sqrt{\pi}\,\Gamma\left(\frac{d-2}{2}+\frac{1}{2}\right)}{\left(\frac{px}{2}\right)^{\frac{d-2}{2}}} J_{\frac{d-2}{2}}(px)$$

$$= \frac{\sqrt{\pi}\,\Gamma\left(\frac{d-1}{2}\right)}{(px)^{\frac{d-2}{2}}}\, 2^{\left(\frac{d-2}{2}\right)} J_{\frac{d}{2}-1}(px) \qquad (1)$$

$$I = \frac{2\pi^{\frac{d-1}{2}}}{\cancel{\Gamma\left(\frac{d-1}{2}\right)}} \frac{1}{(2\pi)^d} \int_0^\infty dp\, p^{d-1} \frac{g^2}{p^2+a^2} \frac{\sqrt{\pi}\, \cancel{\Gamma\left(\frac{d-1}{2}\right)}}{(px)^{\frac{d-2}{2}}} 2^{\frac{d-2}{2}} J_{\frac{d}{2}-1}(px)$$

$$= \frac{2\pi^{\frac{d-1}{2}}}{(2\pi)^d} 2^{\frac{d-2}{2}} \sqrt{\pi} \int_0^\infty dp\, p^{d-1} \frac{g^2}{p^2+a^2} \frac{1}{(px)^{\frac{d-2}{2}}} J_{\frac{d}{2}-1}(px)$$

$$= \frac{2^{\frac{d}{2}} \pi^{\frac{d}{2}}}{(2\pi)^d} \int_0^\infty dp\, p^{d-1} \frac{g^2}{p^2+a^2} \frac{1}{(px)^{(d-2)/2}} J_{\frac{d}{2}-1}(px)$$

$$I = \frac{g^2}{(2\pi)^{d/2}} \int_0^\infty dp\, p^{d-1} \frac{1}{p^2+a^2} \frac{1}{(px)^{(d-2)/2}} J_{\frac{d}{2}-1}(px)$$ ✓

and let $p' = px$

$$I = \frac{g^2}{(2\pi)^{d/2}} \int_0^\infty \frac{dp'}{x} \frac{p'^{\,d-1}}{x^{d-1}} \frac{g^2}{\frac{p'^2}{x^2}+a^2} \frac{1}{p'^{\frac{d-2}{2}}} J_{\frac{d}{2}-1}(p')$$

$$= \frac{g^2}{(2\pi)^{d/2}} \frac{1}{x^{d-2}} \int_0^\infty dp\, p^{d-1-\frac{d}{2}+1} \frac{g^2}{p^2+a^2x^2} J_{\frac{d}{2}-1}(p)$$

$$I = \frac{g^2}{(2\pi)^{\frac{d}{2}}} \frac{1}{x^{d-2}} \int_0^\infty dp\, p^{\frac{d}{2}} \frac{1}{p^2+a^2x^2} J_{\frac{d}{2}-1}(p) \qquad (2)$$

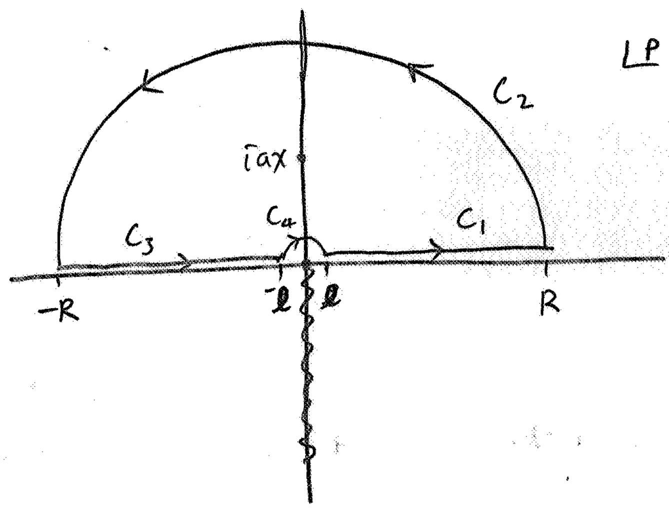
$$I = \frac{g^2}{(2\pi)^{\frac{d}{2}} x^{d-2}} E_2, \quad \text{where } E_2 = \int_0^\infty dp \frac{p^{\frac{d}{2}}}{p^2+a^2x^2} J_{\frac{d}{2}-1}(p)$$

Let $\nu = \frac{d}{2} - 1$, using $J_\nu(p) = \frac{1}{2}\left(H_\nu^{(1)}(p) + H_\nu^{(2)}(p)\right)$,

$$E_2 = \frac{1}{2}\int_0^\infty dp \, \frac{p^{\nu+1}}{p^2 + a^2x^2} H_\nu^{(1)}(p) + \frac{1}{2}\int_0^\infty dp \, \frac{p^{\nu+1}}{p^2+a^2x^2} H_\nu^{(2)}(p) \quad (3)$$

$$= E^{(1)} + E^{(2)}$$

For $E^{(1)}$, take the following contour integral on the branch $-\frac{\pi}{2} < \arg(p) < \frac{3\pi}{2}$, $|p| > 0$ on the complex p-plane



Let $f(p)$ = integrand = $\frac{p^{\nu+1}}{p^2+a^2x^2} H_\nu^{(1)}(p)$,

$$\int_{C_1} f(p)dp + \int_{C_2} f(p)dp + \int_{C_3} f(p)dp + \int_{C_4} f(p)dp = 2\pi i\left(\operatorname*{Res}_{p=iax} f(p)\right)$$

On C_1, $\arg(p) = 0 \Rightarrow p = re^{i0} = r \Rightarrow f(p) = \frac{r^{\nu+1}}{r^2+a^2x^2} H_\nu^{(1)}(r)$

On C_3, $\arg(p) = \pi \Rightarrow p = re^{i\pi} \Rightarrow f(p) = \frac{r^{\nu+1} e^{i\pi(\nu+1)}}{r^2 e^{2\pi i} + a^2x^2} H_\nu^{(1)}(re^{i\pi})$

$$f(p) = \frac{r^{\nu+1} e^{i\pi(\nu+1)}}{r^2 + a^2 x^2} H_\nu^{(1)}(re^{i\pi})$$

By the analytic continuation of $H_\nu^{(1)}$,

$$H_\nu^{(1)}(z e^{i\pi}) = -e^{-\pi i \nu} H_\nu^{(2)}(z) \quad , \; z \in \mathbb{C}$$

$$\Rightarrow f(p) = \frac{r^{\nu+1} e^{i\pi(\nu+1)}}{r^2 + a^2 x^2} (-e^{-\pi i \nu}) H_\nu^{(2)}(z)$$

$$= \frac{-r^{\nu+1} e^{i\pi}}{r^2 + a^2 x^2} H_\nu^{(2)}(z)$$

$$= \frac{r^{\nu+1}}{r^2 + a^2 x^2} H_\nu^{(2)}(z)$$

Also, $\displaystyle\int_{C_3} f(p)\,dp = \int_{-R}^{-\ell} f(-r)\,dr = \int_{\ell}^{R} f(r)\,dr$

$$\Rightarrow \int_{C_1} f(p)dp + \int_{C_3} f(p)\,dp = \int_{\ell}^{R} \frac{r^{\nu+1}}{r^2 + a^2 x^2} H_\nu^{(1)}(r) + \int_{\ell}^{R} \frac{r^{\nu+1}}{r^2 + a^2 x^2} H_\nu^{(2)}(r)$$

$$= 2\int_{\ell}^{R} \frac{r^{\nu+1}}{r^2 + a^2 x^2} J_\nu(r)$$

This is nice, we don't need to worry about evaluating $E^{(2)}$ in (3)

So, $\displaystyle 2\int_{\ell}^{R} \frac{r^{\nu+1}}{r^2 + a^2 x^2} J_\nu(r) = 2\pi i \left(\operatorname*{Res}_{p=iax} f(p)\right) - \int_{C_2} f(p)\,dp - \int_{C_4} f(p)\,dp$

$$\operatorname*{Res}_{p=iax}(f(p)) = \frac{(iax)^{\nu+1} H_\nu^{(1)}(iax)}{2iax} = \frac{1}{2}(iax)^\nu H_\nu^{(1)}(iax)$$

$$= \frac{-i}{2}(ax)^{\nu}\, i^{-\nu+1} H_{\nu}^{(1)}(iax)$$

$$= -\frac{i}{2}(ax)^{\nu} K_{\nu}(ax)\left(\frac{2}{\pi}\right) \Rightarrow 2\pi i\left(\operatorname*{Res}_{p=iax}(f(p))\right) = 2(ax)^{\nu} K_{\nu}(ax)$$

On C_4, $p = le^{i\theta}$, $0 < \theta < \pi$

$$|f(p)| \doteq \left| \frac{(le^{i\theta})^{\nu+1}}{l^2 + a^2x^2} H_{\nu}^{(1)}(le^{i\theta}) \right|$$

$$= \frac{l^{\nu+1}}{l^2 + a^2x^2} \left| H_{\nu}^{(1)}(le^{i\theta}) \right|$$

as $l \to 0$, $H_{\nu}^{(1)}(l) \to l^{-\nu}$

$$\Rightarrow \lim_{l\to 0} |f(l)| \to \frac{l^{\nu+1}}{l^2+a^2x^2} l^{-\nu} \to \frac{l}{a^2x^2} \to 0$$

$$\Rightarrow \left| \int_{C_4} f(p)\,dp \right| \to 0$$

$p \to 0$

For $\int_{C_2} f(p)\,dp$, we note that $H_{\nu}^{(1)}(p) \sim \sqrt{\frac{2}{\pi p}}\, e^{i(p - \frac{\nu\pi}{2} - \frac{\pi}{4})}$

and for large imaginary p $H_{\nu}^{(1)}(p) \to 0$ as $|p| \to \infty$

$$\therefore \quad E_2 = \int_0^{\infty} \frac{r^{\nu+1}}{r^2+a^2x^2} J_{\nu}(r) = \frac{1}{2}(2\pi i)\left\{ \operatorname*{Res}_{p=iax}(f(p)) \right\}$$

$$= (ax)^{\nu} K_{\nu}(ax)$$

$$\Rightarrow I = \frac{g^2}{(2\pi)^{\frac{d}{2}} x^{d-2}} (ax)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(ax) = \frac{g^2}{(2\pi)^{\frac{d}{2}}} \left(\frac{a}{x}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(ax)$$

$$\therefore \boxed{\int \frac{d^d p}{(2\pi)^d} \, e^{i\vec{p}\cdot\vec{x}} \, \frac{g^2}{p^2+a^2} = \frac{g^2}{(2\pi)^{\frac{d}{2}}} \left(\frac{a}{x}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(ax)} \qquad (4)$$

For $d=1$, From (4) we get

$$\int \frac{dp}{2\pi} e^{ipx} \frac{g^2}{p^2+a^2} = \frac{g^2}{\sqrt{2\pi}} \sqrt{\frac{x}{a}} \, K_{-\frac{1}{2}}(ax)$$

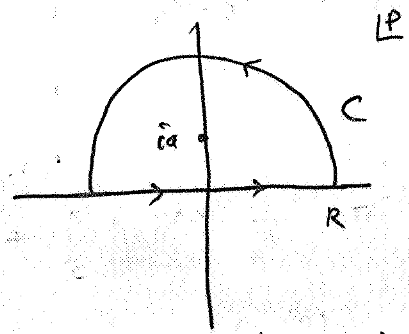
$$= \frac{g^2}{\sqrt{2\pi}} \sqrt{\frac{x}{a}} \, e^{-ax} \sqrt{\frac{\pi}{2ax}}$$

$$= \frac{g^2}{2} \frac{1}{a} e^{-ax}$$

If we did the contour integral directly, we will get

$$\int \frac{dp}{2\pi} e^{ipx} \frac{g^2}{p^2+a^2} = \frac{g^2}{2\pi} \int_{-\infty}^{\infty} dp \, e^{ipx} \frac{1}{p^2+a^2}$$

Note that the limit of integration now goes from $-\infty$ to ∞ because we're in 1D!



$$\int_{-\infty}^{\infty} f(p)\,dp + \underbrace{\cancel{\int_C f(p)\,dp}}_{\to 0 \text{ as } R\to\infty} = 2\pi i \left(\operatorname*{Res}_{p=ia} f(p) \right), \quad f(p) = \frac{e^{ipx}}{p^2+a^2}$$

$$\Rightarrow \int_{-\infty}^{\infty} dp \, \frac{e^{ipx}}{p^2+a^2} = 2\pi i \, \frac{e^{-ax}}{2ia} = \pi \frac{e^{-ax}}{a}$$

$$\Rightarrow \quad I = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx} \frac{g^2}{p^2+a^2} = \frac{g^2}{2\pi} \frac{\pi e^{-ax}}{a}$$

$$= \frac{g^2}{2} \frac{1}{a} e^{-ax} \quad \checkmark$$

In $d=3$, From (4),

$$I = \frac{g^2}{(2\pi)^{3/2}} \left(\frac{a}{x}\right)^{\frac{1}{2}} K_{\frac{1}{2}}(ax)$$

$$= \frac{g^2}{(2\pi)^{3/2}} \left(\frac{a}{x}\right)^{\frac{1}{2}} \sqrt{\frac{\pi}{2ax}} \, e^{-ax}$$

$$= \frac{g^2}{4\pi} \frac{1}{x} e^{-ax} \quad \checkmark$$

as we showed in 1. (a).

In the large x limit, $K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} \, e^{-z}$

$$\Rightarrow \quad I \sim \left(\frac{a}{x}\right)^{\frac{d}{2}-1} \left(\frac{1}{ax}\right)^{\frac{1}{2}} e^{-ax} \sim \frac{a^{\frac{d-3}{2}}}{x^{(d-1)/2}} e^{-ax}$$

And as $x \to 0$, $K_\nu(z) \sim \left(\frac{z}{2}\right)^{-\nu}$

$$\Rightarrow \quad I \sim \left(\frac{a}{x}\right)^{\frac{d}{2}-1} \left(\frac{ax}{2}\right)^{-\frac{d}{2}+1} \sim \frac{1}{x^{d-2}}$$

2.

(a)

(i) $V(r) = Ae^{-ar}$

In the first Born approximation

$$f_k(\theta,\phi) = -\frac{m}{2\pi}\tilde{V}(\vec{Q})$$

First let's find $\tilde{V}$ for $V(r) = Ae^{-ar}$.

$$\tilde{V}(\vec{Q}) = \int d^3r \; e^{-i\vec{Q}\cdot\vec{r}} \, Ae^{-ar} \quad , \; r = |\vec{r}|$$

$$= 2\pi A \int_0^\infty r^2 dr \, e^{-ar} \int_{-1}^{1} d\cos\theta \; e^{-iQr\cos\theta}$$

$$= 2\pi A \int_0^\infty r^2 dr \, e^{-ar} \frac{i}{Qr}\left(e^{-iQr} - e^{iQr}\right)$$

$$= \frac{2\pi A i}{Q} \int_0^\infty r\, dr \, e^{-ar}\left(e^{-iQr} - e^{iQr}\right)$$

$$\| \quad re^{-ar} = -\frac{d}{da} e^{-ar} \quad \|$$

$$= -\frac{2\pi Ai}{Q}\frac{d}{da}\left\{\int_0^\infty dr \; e^{r(-a-iQ)} - \int_0^\infty dr \; e^{r(-a+iQ)}\right\}$$

$$= \frac{-2\pi Ai}{Q}\frac{d}{da}\left\{\frac{1}{-a-iQ} e^{r(-a-iQ)}\Big|_0^\infty - \frac{1}{-a+iQ} e^{r(-a+iQ)}\Big|_0^\infty\right\}$$

$$= \frac{-2\pi Ai}{Q}\frac{d}{da}\left\{\frac{1}{a+iQ} + \frac{1}{-a+iQ}\right\}$$

$$= \frac{-2\pi A i}{Q} \frac{d}{da} \left\{ \frac{-a+iQ+a+iQ}{-a^2-Q^2} \right\}$$

$$= \frac{+2\pi A i}{Q} \, 2iQ \, \frac{d}{da} \frac{1}{a^2+Q^2}$$

$$= -4\pi A \frac{-2a}{(a^2+Q^2)^2}$$

$$= 8\pi A \frac{a}{(a^2+Q^2)^2}$$

So, $f_k(\theta,\phi) = -\frac{m}{2\pi} \tilde{V}(\vec{Q})$

$$= -\frac{m}{2\pi} 8\pi A \frac{a}{(a^2+Q^2)^2}$$

$$= -4mA \frac{a}{(a^2+Q^2)^2}$$

$$\frac{d\sigma}{d\Omega} = 16 m^2 A^2 \frac{a^2}{(a^2+Q^2)^4}$$

$$\frac{d\sigma}{d\cos\theta} = \frac{32\pi \, m^2 A^2 a^2}{(a^2+Q^2)^4} \quad , \quad Q^2 = |\vec{Q}|^2 = |k(\hat{r}-\hat{z})|^2$$

$$= 4k^2 \sin^2\frac{\theta}{2} \quad , k^2 = 2mE$$

$$= 8mE \sin^2\frac{\theta}{2}$$

$$\frac{d\sigma}{d\cos\theta} = \frac{32\pi \, m^2 A^2 a^2}{(a^2 + 8mE\sin^2\frac{\theta}{2})^4} \quad // \qquad (1)$$

(ii) $V(r) = \begin{cases} B, & r<a \\ 0, & r>a \end{cases} = B\,\theta(a-r)$, where

$$\theta(x) = \begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases}$$

$$\tilde{V}(Q) = \int d^3r \; e^{-i\vec{Q}\cdot\vec{r}} \; B\,\theta(a-r)$$

$$= B(2\pi) \int r^2 dr \; \theta(a-r) \int_{-1}^{1} d(\cos\theta) \; e^{-iQr\cos\theta}$$

$$= 2\pi B \int_0^\infty r^2 dr \; \theta(a-r) \frac{i}{Qr} \left(e^{-iQr} - e^{iQr} \right)$$

$$= \frac{2\pi i B}{Q} \int_0^a r\,dr \left(e^{-iQr} - e^{+iQr} \right)$$

$$= \frac{2\pi i B}{Q} \left\{ i \frac{d}{dQ} \int_0^a dr\; e^{-iQr} + i \frac{d}{dQ} \int_0^a dr\; e^{iQr} \right\}$$

$$= -\frac{2\pi B}{Q} \frac{d}{dQ} \left\{ \frac{i}{Q} e^{-iQr}\Big|_0^a - \frac{i}{Q} e^{iQr}\Big|_0^a \right\}$$

$$= -\frac{2\pi B}{Q} \frac{d}{dQ} \left\{ \frac{i}{Q} \left(e^{-iQa} - 1 - e^{iQa} + 1 \right) \right\}$$

$$= +\frac{2\pi B i}{Q} \frac{d}{dQ} \left(\frac{e^{iaQ} - e^{-iaQ}}{Q} \right)$$

$$= -\frac{4\pi B}{Q} \frac{d}{dQ} \left(\frac{\sin(aQ)}{Q} \right)$$

$$= -\frac{4\pi B}{Q^3} \left(Qa\cos aQ - \sin aQ \right) \quad //$$

$$\Rightarrow f_k(\theta,\phi) = -\frac{m}{2\pi}\left(-\frac{4\pi B}{Q^2}\left(Qa\cos aQ - \sin aQ\right)\right)$$

$$= \frac{2mB}{Q^3}\left(Qa\cos aQ - \sin aQ\right)$$

$$\frac{d\sigma}{d\Omega} = \frac{4m^2B^2}{Q^6}\left(Qa\cos aQ - \sin aQ\right)^2$$

$$\frac{d\sigma}{d\cos\theta} = \frac{8\pi m^2 B^2}{\left(2\sqrt{2mE}\sin\frac{\theta}{2}\right)^6}\left[2a\sqrt{2mE}\sin\frac{\theta}{2}\cos\left(2a\sqrt{2mE}\sin\frac{\theta}{2}\right) - \sin\left(2a\sqrt{2mE}\sin\frac{\theta}{2}\right)\right]^2 \qquad (2)$$

For Yukawa potential, $V(r) = \frac{g^2}{4\pi r}e^{-ar} = \frac{c}{r}e^{-ar}$, $c = \frac{g^2}{4\pi}$

$$\tilde{V}(\vec{Q}) = \frac{g^2}{Q^2+a^2}$$

$$f_k = -\frac{m}{2\pi}\cdot\frac{g^2}{Q^2+a^2}$$

$$\frac{d\sigma}{d\cos\theta} = 2\pi\,\frac{m^2}{4\pi^2}\,\frac{g^4}{(Q^2+a^2)^2} = \frac{m^2}{2\pi}\,\frac{g^4}{(Q^2+a^2)^2}$$

$$\frac{d\sigma}{d\cos\theta} = 8\pi c^2 m^2 \frac{1}{\left(8mE\sin^2\frac{\theta}{2}+a^2\right)^2} \qquad (3)$$

For $a = m = k = 1$ we can compare the three results displayed in eqns (1), (2), and (3)

(b)

(i) $\rho(r) = A e^{-ar}$

$$\int d^3r\, \rho(r) = e$$

$$\Rightarrow e = A \int d^3r\, e^{-ar} = A \int d^2\Omega \int_0^\infty r^2 dr\, e^{-ar}$$

$$= A\, \mathrm{Vol}(S^2) \frac{d^2}{da^2} \int_0^\infty dr\, e^{-ar}$$

$$= A (4\pi) \frac{d^2}{da^2} \left(+ \frac{1}{a} \right)$$

$$e = A (4\pi) \frac{2}{a^3}$$

$$\Rightarrow A = \frac{a^3 e}{8\pi} \quad //$$

$$f_k(\theta, \phi) = -\frac{m}{2\pi} \tilde{V}(\vec{Q}) = \frac{-m q_1}{2\pi \epsilon_0 |\vec{Q}|^2} \tilde{\rho}(Q) \quad , \quad q_1 = e$$

$$\tilde{\rho}(Q) = 8\pi A \frac{a}{(a^2 + Q^2)^2} \quad \text{from } 1(a).$$

$$f_k = -\frac{m}{2\pi} \frac{e}{\epsilon_0 |\vec{Q}|^2} 8\pi A \frac{a}{(a^2 + |\vec{Q}|^2)^2}$$

$$= \frac{-m e^2}{2\pi \epsilon_0} \frac{a^4}{|\vec{Q}|^2 (a^2 + |\vec{Q}|^2)^2}$$

$$\frac{d\sigma}{d\cos\theta} = \frac{m^2 e^4}{2\pi\epsilon_0^2} \frac{a^8}{Q^4(a^2+Q^2)^4} \qquad (1)$$

(ii) $\rho(r) = \begin{cases} B, & r<a \\ 0, & r>a \end{cases} = B\,\theta(a-r)$

$$\int d^3r\,\rho(r) = e$$

$$\Rightarrow e = 4\pi\int_0^\infty r^2 dr\; B\,\theta(a-r)$$

$$= B\,4\pi\int_0^a r^2 dr$$

$$e = 4\pi B\frac{a^3}{3}$$

$$\Rightarrow B = \frac{3}{4\pi}\frac{e}{a^3}$$

From 2a (ii),

$$\tilde{\rho}(\vec{Q}) = -\frac{4\pi B}{Q^3}(Qa\cos aQ - \sin aQ)$$

$$f_k = -\frac{me}{2\pi\epsilon_0}\frac{1}{Q^2}\left[-\frac{4\pi B}{Q^3}(Qa\cos aQ - \sin aQ)\right]$$

$$= \frac{me^2}{2\pi\epsilon_0 a^3}\frac{3}{Q^5}(Qa\cos aQ - \sin aQ)$$

$$\frac{d\sigma}{d\cos\theta} = \frac{m^2e^4}{2\pi\epsilon_0^2 a^6}\frac{9}{Q^{10}}(Qa\cos aQ - \sin aQ)^2 \qquad (2)$$

For a scattering off of a pure Coulomb potential we take $a=0$ in eqn. (3) of 2 (a).

$$\frac{d\sigma}{d\cos\theta} = \frac{8\pi c^2 \cancel{m^2}}{8^2 \cancel{m^2} E^2 \sin^4\frac{\theta}{2}} = \frac{\pi c^2}{8E^2\sin^4\frac{\theta}{2}}$$

$$\parallel \quad c = \frac{g^2}{4\pi} = \frac{e^2}{4\pi\epsilon_0}$$

$$= \frac{\pi e^4}{8\times 4^2\pi^2\epsilon_0^2 E^2\sin^4\frac{\theta}{2}}$$

$$= \frac{e^4}{128\pi\epsilon_0^2 E^2 \sin^4\frac{\theta}{2}} \qquad (3)$$

Plugging $Q^2 = 8mE\sin^2\frac{\theta}{2}$ in (1) & (2), we have

(i) $$\frac{d\sigma}{d\cos\theta} = \frac{e^4}{128\pi\epsilon_0^2}\frac{a^8}{E^2\sin^4\frac{\theta}{2}}\frac{1}{(a^2+8mE\sin^2\frac{\theta}{2})^4}$$

(ii) $$\frac{d\sigma}{d\cos\theta} = \frac{9e^4m^2}{2\pi\epsilon_0^2 a^6}\frac{1}{(8mE\sin^2\frac{\theta}{2})^5}\left[2a\sqrt{2mE}\sin\frac{\theta}{2}\cos\left(2a\sqrt{2mE}\sin\frac{\theta}{2}\right) - \sin\left(2a\sqrt{2mE}\sin\frac{\theta}{2}\right)\right]^2$$

Coulomb, $$\frac{d\sigma}{d\cos\theta} = \frac{e^4}{128\pi\epsilon_0^2}\frac{1}{E^2\sin^4\frac{\theta}{2}}$$

(c)

$$\rho(r) = e\,e^{-ar} - \frac{e}{8} e^{-ar/2}$$

$$= \tilde{A} e^{-ar} - \tilde{B} e^{-br}, \quad \tilde{A} = e, \quad \tilde{B} = \frac{e}{8}, \quad b = \frac{a}{2}$$

From (b)

$$\tilde{\rho}(Q) = 8\pi \tilde{A} \frac{a}{(a^2+Q^2)^2} - 8\pi \tilde{B} \frac{b}{(b^2+Q^2)^2}$$

$$F(Q) = 8\pi \frac{a}{(a^2+Q^2)^2} - \pi \frac{b}{(b^2+Q^2)^2}$$

$$\frac{d\sigma}{d\cos\theta} = \frac{e^4}{128\pi E^2 \sin^4\frac{\theta}{2}} \pi^2 \left(\frac{8a}{(a^2+Q^2)^2} - \frac{b}{(b^2+Q^2)^2} \right)^2$$

$$= \frac{e^4\pi}{128\, E^2 \sin^4\frac{\theta}{2}} \left(\frac{8a}{(a^2+8mE\sin^2\frac{\theta}{2})^2} - \frac{b}{(b^2+8mE\sin^2\frac{\theta}{2})^2} \right)^2$$

For small Q, $F(Q) \sim \left(\frac{8}{a^3} - \frac{1}{b^3}\right) - 2Q^2\left(\frac{8}{a^5} - \frac{1}{b^5}\right) + O(Q^4)$

When total charge is 0, $b = \frac{a}{2}$, otherwise $b \neq \frac{a}{2}$.

So for 0 total charge, $F(Q) \sim Q^2$, so $F(0) = 0$ //

But, when total charge $\neq 0$, i.e. $b \neq \frac{a}{2}$, then $F(0)$ is finite.

This is also reflected in the differential cross-section:

$$\frac{d\sigma}{d\cos\theta} \sim \frac{1}{Q^2} |F(Q)|^2$$

For $b \neq \frac{a}{2}$, $\frac{d\sigma}{d\cos\theta} \sim \frac{1}{Q^2}$ which diverges for $Q \to 0$. (Forward scattering). However for $b = \frac{a}{2}$, $\frac{d\sigma}{d\cos\theta} \sim \frac{1}{Q^2}(Q^2 + O(Q^4)) \sim$ finite for $Q \to 0$.

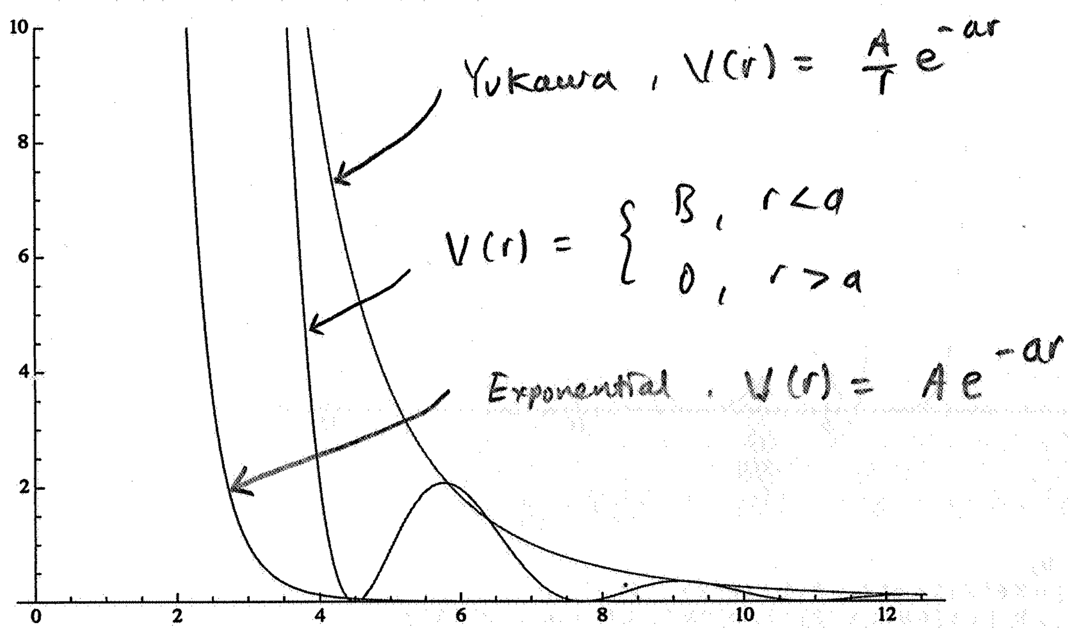
2. a.

Let $t = 4 \sin^2(\frac{\theta}{2})$

In[1035]:=
```
Clear[A, B, CC]
sig1[t_] = 32 Pi A^2 / (1 + t^2)^4;
sig2[t_] = 8 Pi B^2 * (t Cos[t] - Sin[t])^2 / t^6;
sigYuk[t_] = 8 Pi CC^2 / (t^2 + 1)^2;
```

In[1039]:=
```
A = 10;
B = 10;
CC = 10;
Plot[{sig1[t], sig2[t], sigYuk[t]}, {t, 0, 4 Pi}, PlotRange → {0, 10}]
```

Out[1042]=

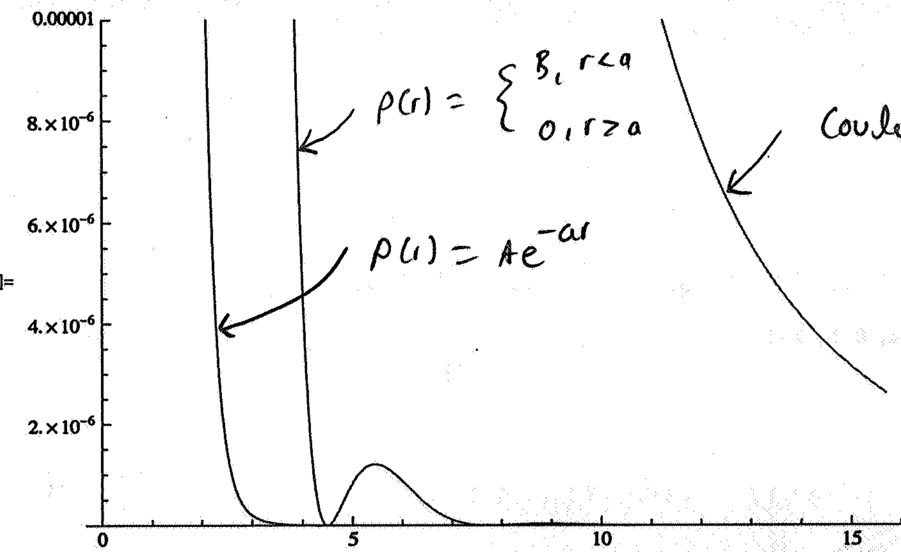


Notice the distinctive feature of the sharp edge potential. Experimentally we could look for such zeros by doing a scattering experiment and looking for zeros in the scattering cross-section along some direction.

2. b.

```
sig2a[t_] = 1 / (2 Pi t^4) * 1 / (1 + t^2) ^4;
sig2b[t_] = 9 / (2 Pi t^10) * (t * Cos[t] - Sin[t]) ^2;
sigCoul[t_] = 1 / (2 Pi t^4);
Plot[{sig2a[t], sig2b[t], sigCoul[t]}, {t, 0, 5 Pi}, PlotRange → {0, 0.00001}]
```

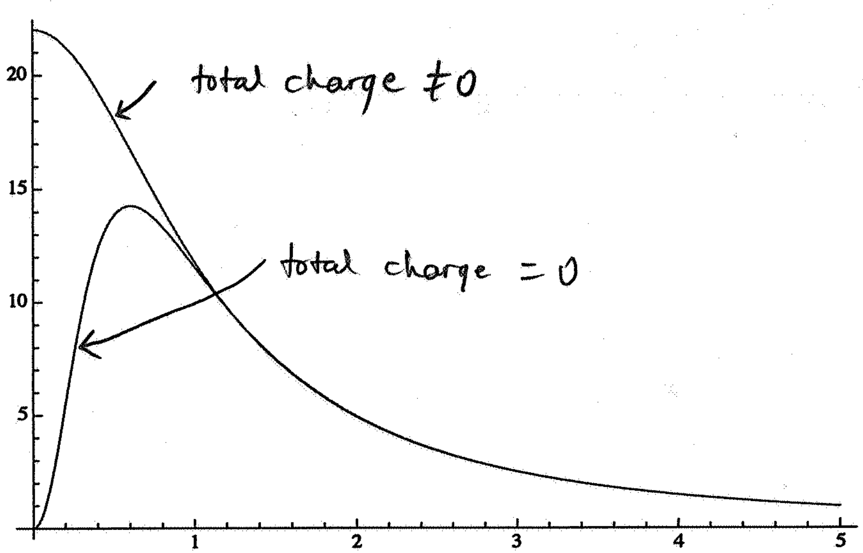
Out[1034]=



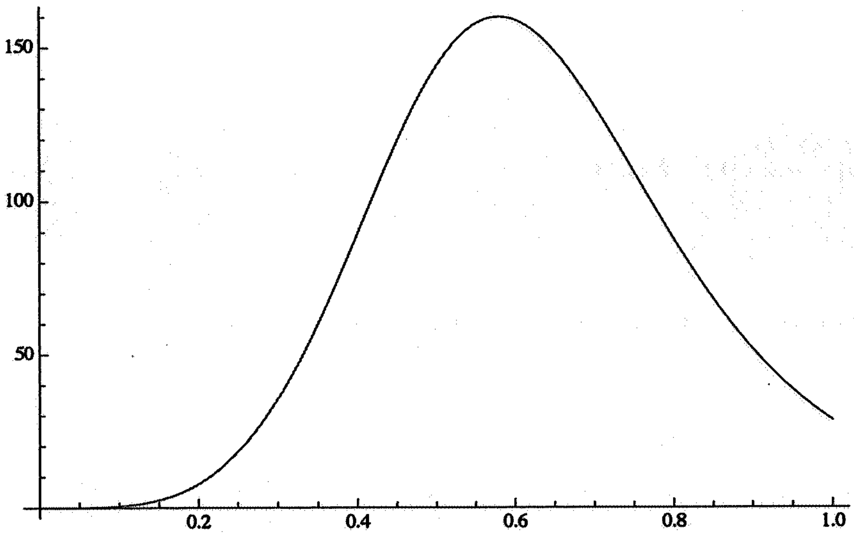
2. c.

```
Clear[q, b]
f[q_, b_] = 8 Pi / (1 + q^2) - Pi b / (b^2 + q^2) ^2;
sigNue[t_, b_] = f[4 Sin[t / 2] ^2, b] ^2 / (32 Pi Sin[t / 2] ^4 );

Plot[{f[q, 1 / 2], f[q, 1]}, {q, 0, 5}]
```



```
Plot[{sigNue[t, 1/2]}, {t, 0, 1}]
```




```
Plot[{sigNue[t, 1]}, {t, 0.5, 1}]
```