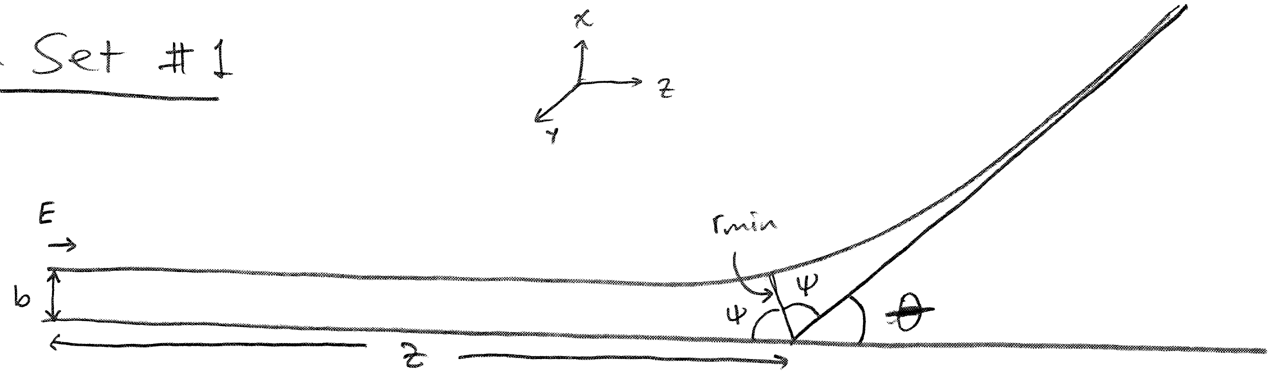
Solution Set #1

1.



- There is no friction, so total energy is a constant of motion
- There is no torque, so angular momentum is also a constant of motion.

$$E_{tot} = E_{Kin} + V(r)$$

$$= \frac{1}{2} m \vec{v}^2 + \frac{k}{r} \quad , \quad k = \frac{q_1 q_2}{4\pi\epsilon_0}$$

$$\vec{l} = \vec{r} \times \vec{p} = (z\hat{z} + b\hat{x}) \times m(v_z \hat{z} + v_x \hat{x})$$

The particle is incident along z-direction, so far away from the origin $\vec{v} = v_0 \hat{z}$

$$\Rightarrow \boxed{\vec{l} = -mbv_0\,\hat{y}} = \text{constant.} \qquad (1)$$

Also, for $r \to \infty$ $V(r) \to 0$ $\Rightarrow E_{tot} = E = \frac{1}{2} m v_0^2$

$$\Rightarrow v_0 = \sqrt{2E/m}$$

$$l = |\vec{l}| = b\sqrt{2mE}$$

$$E_{total} - E_{kin} - V(r) = 0 \qquad (2)$$

$$\Rightarrow E - \frac{1}{2} m v^2 - V(r) = 0 \quad , \quad \vec{v} = \frac{d\vec{s}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}$$

$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$v^2 = \vec{v}\cdot\vec{v} = \dot{r}^2 + r^2\dot{\theta}^2$$

$$= \dot{r}^2 + \frac{l^2}{m^2 r^2} \quad , \text{since } l = mr^2\dot{\theta}$$

$$\Rightarrow E - \frac{1}{2}m\left(\dot{r}^2 + \frac{\ell^2}{m^2r^2}\right) - V(r) = 0$$

$$\frac{2}{m}\left(E - V(r)\right) - \frac{\ell^2}{m^2r^2} = \dot{r}^2$$

$$\left\| \; \dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = r'\dot{\theta} = r'\frac{\ell}{mr^2} \; \right\|$$

$$\Rightarrow \frac{2}{m}\left(E - V(r)\right) - \frac{\ell^2}{m^2r^2} = (r')^2\frac{\ell^2}{m^2r^4}$$

$$\boxed{r' = \frac{dr}{d\theta} = \sqrt{\frac{2mr^4}{\ell^2}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]}} \qquad (3)$$

Change variables to $u = \frac{1}{r}$ $\Rightarrow \frac{dr}{d\theta} = \frac{dr}{du}\frac{du}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$

$$(3) \Rightarrow -\frac{du}{d\theta} = u^2\sqrt{\frac{2m}{\ell^2}\frac{1}{u^4}\left[E - V\left(\tfrac{1}{u}\right) - \frac{\ell^2}{2m}u^2\right]} \quad , V\left(\tfrac{1}{u}\right) = ku$$

$$-\frac{du}{d\theta} = \sqrt{\frac{2m}{\ell^2}\left[E - ku - \frac{\ell^2}{2m}u^2\right]}$$

$$\int d\theta = -\int \frac{du}{\sqrt{\frac{2m}{\ell^2}\left[E - ku - \frac{\ell^2}{2m}u^2\right]}}$$

$$\theta - \theta' = -\int \frac{du}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mk}{\ell^2}u - u^2}}$$; θ' is integration constant.

The integral has the general form.

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-a}} \arccos\left(-\frac{b+2ax}{\sqrt{b^2-4ac}}\right)$$

$$\Rightarrow \theta-\theta' = -\arccos\left(-\frac{-\frac{2mk}{l^2}+2u}{\sqrt{\frac{4m^2k^2}{l^4}+\frac{8mE}{l^2}}}\right) \qquad (4)$$

$$\Rightarrow \cos(\theta-\theta') = \left(\frac{mk}{l^2}+u\right)\frac{1}{\sqrt{1+\frac{2El^2}{mk^2}}}\left(\frac{l^2}{mk}\right)$$

$$\Rightarrow u = \frac{mk}{l^2}\left(-1+\sqrt{1+\frac{2El^2}{mk^2}}\cos(\theta-\theta')\right)$$

$$\Rightarrow \boxed{\frac{1}{r} = \frac{mk}{l^2}\left(\epsilon\cos(\theta-\theta')-1\right), \quad \epsilon = \sqrt{1+\frac{2El^2}{mk^2}}}$$

Boundary conditions:

$\frac{1}{r}$ is maximum when $\cos(\theta-\theta') = 1$. If we set $\theta=0$ to be the turning point, or the point of closest approach, then we have to set $\theta'=0$.

We get $\frac{1}{r} = \frac{mk}{l^2}(\epsilon\cos\theta - 1)$.

ψ is determind by letting $r\to\infty$

$$\Rightarrow 0 = \epsilon\cos\psi - 1 \quad \Rightarrow \quad \cos\psi = \frac{1}{\epsilon}.$$

Since $2\psi+\theta=\pi$, $\Rightarrow \psi = \frac{\pi}{2}-\frac{\theta}{2}$ $\Rightarrow \cos\psi = \sin\frac{\theta}{2}$

$$\Rightarrow \quad \sin\frac{\theta}{2} = \frac{1}{\epsilon}$$

$$\epsilon^2 = \frac{1}{\sin^2\theta/2} = 1 + \cot^2\frac{\theta}{2}$$

$$\cancel{1} + \frac{2El^2}{mk^2} = \cancel{1} + \cot^2\frac{\theta}{2}$$

$$l^2 = 2mEb^2$$

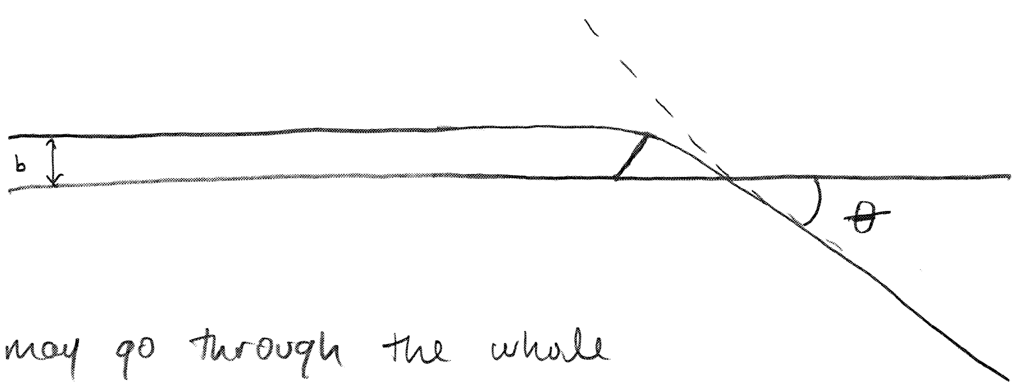
$$\Rightarrow \quad \frac{4E^2b^2}{k^2} = \cot^2\frac{\theta}{2}$$

$$\frac{2Eb}{k} = \cot\frac{\theta}{2}$$

$$b = \frac{k}{2E}\cot\frac{\theta}{2}$$

$$\boxed{b = \frac{q_1 q_2}{8\pi\epsilon_0 E}\cot\frac{\theta}{2}}$$

b.



We may go through the whole calculation, ~~but~~ we can easily observe that $\theta \to -\theta$ and $k \to -k$ are the only differences b/w attractive and repulsive potentials. Therefore, the formula for b remains the same.

c. $$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \qquad (1)$$

$$d\Omega = \sin\theta\, d\theta\, d\phi = -d(\cos\theta)\, d\phi$$

RHS of (1) has no ϕ dependence by azimuthal symmetry

$$\Rightarrow \int d\phi \ (1) \rightarrow -\frac{d\sigma}{d(\cos\theta)} = \frac{2\pi b}{\sin\theta}\left|\frac{db}{d\theta}\right| \qquad (2)$$

$$\frac{db}{d\theta} = -\frac{q_1 q_2}{16\pi\epsilon_0 E}\csc^2\left(\frac{\theta}{2}\right)$$

$$\Rightarrow -\frac{d\sigma}{d(\cos\theta)} = 2\pi\left(\frac{q_1 q_2}{8\pi\epsilon_0 E}\cot\frac{\theta}{2}\right)\frac{1}{\sin\theta}\ \frac{q_1 q_2}{16\pi\epsilon_0 E}\ \frac{1}{\sin^2\frac{\theta}{2}}$$

$$= \frac{q_1^2 q_2^2}{64\pi\epsilon_0^2 E^2}\ \frac{\cos\frac{\theta}{2}}{\sin^3\frac{\theta}{2}\sin\theta} = \frac{q_1^2 q_2^2}{64\pi\epsilon_0^2 E^2}\ \frac{\cos\frac{\theta}{2}}{\sin^3\frac{\theta}{2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}$$

$$\boxed{-\frac{d\sigma}{d\cos\theta} = \frac{q_1^2 q_2^2}{128\pi\epsilon_0^2 E^2}\ \frac{1}{\sin^4\frac{\theta}{2}}}$$

d. let $x = \cos\theta = 1 - 2\sin^2\frac{\theta}{2}$

$$\Rightarrow \sin^4\frac{\theta}{2} = \frac{1}{4}(1-x)^2 \quad //$$

$$\sigma = \int_{-1}^{1} d\cos\theta \, \frac{q_1^2 q_2^2}{128\pi\epsilon_0^2 E^2} \frac{1}{\sin^4\frac{\theta}{2}}$$

$$= \frac{q_1^2 q_2^2}{128\pi\epsilon_0^2 E^2} 4\int_{-1}^{1} dx \, (1-x)^{-2}$$, let $y = 1-x \Rightarrow dx = -dy$

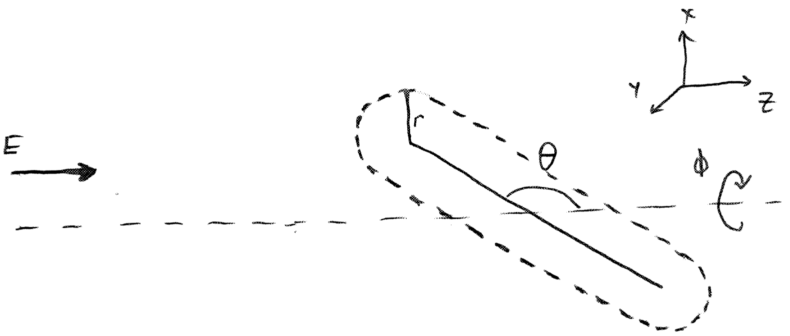
$$= \frac{q_1^2 q_2^2}{32\pi\epsilon_0^2 E^2} \int_0^2 dy \, \frac{1}{y^2} ,$$

which diverges like $\frac{1}{y}$ as $y \to 0$

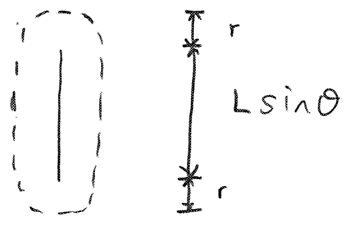
The infinite contribution to σ comes from scattering at small angle $\Rightarrow$ large impact parameter. This results from the infinite range of coulomb interaction. All the ∞ cross-section is saying is that there will be scattering regardless of the impact parameter, and we should not be concerned about this.

Further, all bare charges are eventually screened. For example, for distances larger than atomic radii atoms look neutral, and even bare ions in space get screened by plasma of electrons in space. Therefore, above some distance scale the scattering potential gets cut off and the cross-section turns out finite.

2.



Side view



Front view

a. $\sigma = \frac{1}{2}\pi r^2 + \frac{1}{2}\pi r^2 + 2rL\sin\theta$

$$\sigma(\theta) = \pi r^2 + 2Lr\sin\theta$$

b. We average over all possible orientations for the molecule. Each small angle $d\Omega$ is equally likely to occur. The probability to be within a small solid angle $d\Omega$ of the point (θ, ϕ) is given by the ratio of the "volume" of $d\Omega$ and the volume of the whole space (S^2).

$$= \frac{1}{Vol(S^2)} d\Omega = \frac{1}{4\pi} \sin\theta \, d\theta \, d\phi .$$

$$\sigma_{av} = \int d\Omega \, \sigma(\theta, \phi)$$

$$= \frac{1}{4\pi}\int_{-1}^{1} d\cos\theta \, d\phi \, (\pi r^2 + 2Lr\sin\theta)$$

$$= \frac{1}{4\pi}(2\pi)\left[2\pi r^2 + 2Lr\int_{-1}^{1} dx \sqrt{1-x^2}\right. , \quad x = \cos\theta$$

$$= \pi r^2 + Lr\left(\frac{\pi}{2}\right)$$

$$\boxed{\sigma_{av} = \pi r^2 + \frac{\pi L r}{2}}$$

c. Now, the probability density for the orientation of the molecule is no longer uniform $\frac{1}{4\pi}$, but instead given by $|\langle\theta,\phi|\psi\rangle|^2 = |Y_{lm}(\theta,\phi)|^2$ for a state $|\psi\rangle$ that is an eigenstate of L and L_z.

$$\therefore \sigma = \int d\Omega \, |Y_{lm}|^2 \sigma(\theta,\phi)$$

$\sigma(\theta,\phi)$ will remain the same as in (a) though!

$$\sigma = \int d\Omega \, |Y_{lm}|^2 (\pi r^2 + 2Lr\sin\theta) \quad , \quad \int d\Omega |Y_{lm}|^2 = 1$$

$$= \pi r^2 + 2Lr \int d\Omega \, |Y_{lm}|^2 \sin\theta ,$$

$Y_{lm}(\theta,\phi) \sim e^{im\phi} \Rightarrow |Y_{lm}|^2 \sim e^{im\phi}e^{-im\phi}$, independent of ϕ

$$\sigma_{lm} = \pi r^2 + 4\pi L r \int d\cos\theta \, |Y_{lm}|^2 \sin\theta$$

$$\sigma_{00} = \pi r^2 + 4\pi L r \int d\cos\theta \, |Y_{00}|^2 \sin\theta \quad , \quad Y_{00} = \sqrt{\frac{1}{4\pi}}$$

$$= \pi r^2 + Lr \int d\cos\theta \, \sin\theta$$

$$\boxed{\sigma_{00} = \pi r^2 + \frac{\pi L r}{2}}$$

- $\sigma_{10} = \pi r^2 + 4\pi L r \frac{3}{4\pi} \int d\cos\theta \ \sin\theta \cos^2\theta$

$$= \pi r^2 + 3Lr \int_{-1}^{1} dx \ x^2\sqrt{1-x^2} = \pi r^2 + 3Lr\left(\frac{\pi}{8}\right)$$

$$\boxed{\sigma_{10} = \pi r^2 + \frac{3\pi}{8} Lr}$$

$\int dx \left(\sqrt{1-x^2}\right)^3$

- $\sigma_{1-1} = \sigma_{11} = \pi r^2 + 4\pi L r \frac{3}{8\pi} \int d\cos\theta \ \sin^3\theta$

$$= \pi r^2 + \frac{3Lr}{2}\left(\frac{3\pi}{8}\right)$$

$$\boxed{\sigma_{1-1} = \sigma_{11} = \pi r^2 + \frac{9\pi}{16} Lr}$$

- $\sigma_{20} = \pi r^2 + 4\pi L r \left(\frac{5}{16\pi}\right) \int d\cos\theta \ \sin\theta \left(3\cos^2\theta - 1\right)^2$

$$= \pi r^2 + \frac{5}{4} Lr \int dx \ \sqrt{1-x^2}\left(3x^2-1\right)^2$$

$$= \pi r^2 + \frac{5}{4} Lr \left(\frac{5\pi}{16}\right)$$

$$\boxed{\sigma_{20} = \pi r^2 + \frac{25\pi}{64} Lr}$$

- $\sigma_{2-1} = \sigma_{21} = \pi r^2 + 4\pi L r\left(\frac{15}{8\pi}\right) \int d\cos\theta \ \sin^3\theta \cos^2\theta$

$$= \pi r^2 + \frac{15}{2} Lr \left(\frac{\pi}{16}\right)$$

$$\boxed{\sigma_{2-1} = \sigma_{21} = \pi r^2 + \frac{15\pi}{32} Lr}$$

- $\sigma_{2-2} = \sigma_{22} = \pi r^2 + 4\pi L r \left(\frac{15}{32\pi}\right) \int d\cos\theta \ \sin^5\theta = \pi r^2 + \frac{15}{8} Lr \left(\frac{5\pi}{16}\right)$

$$\boxed{\sigma_{2-2} = \sigma_{22} = \pi r^2 + \frac{75\pi}{128} Lr}$$

d) $\sigma_l = \frac{1}{2l+1} \sum_{m} \sigma_{lm}$

$$\sigma_0 = \pi r^2 + \frac{\pi}{2} L r$$

$$\sigma_1 = \frac{1}{3}\left\{ 3\pi r^2 + L r \pi \left(\frac{3}{8} + \frac{9}{8} \right) \right\}$$

$$= \pi r^2 + \frac{L r \pi}{2}$$

$$\sigma_2 = \frac{1}{5}\left\{ 5\pi r^2 + L r \pi \left(\frac{25}{64} + \frac{15}{16} + \frac{75}{64} \right) \right\}$$

$$= \pi r^2 + \frac{L r \pi}{64}\left(5 + 12 + 15 \right)$$

$$= \pi r^2 + \frac{L r}{2} \pi$$

The σ_l's are all the same. This result holds for all l's. The proof is easy.

$$\left(\frac{1}{2l+1}\right) \sum_{m} \int d\Omega \, |Y_{lm}|^2 \sin\theta$$

$$= \int d\Omega \sin\theta \left\{ \frac{1}{2l+1} \sum_{m} |Y_{lm}|^2 \right\}$$

$$= \int d\Omega \sin\theta \left(\frac{1}{2l+1} \times \frac{2l+1}{4\pi} \right)$$

$$= \frac{1}{4\pi} \int d\Omega \sin\theta$$

$$= \frac{2\pi}{4\pi} \int d\cos\theta \sin\theta = \frac{\pi}{4}$$