Multipole Transitions and Selection Rules

Up to this point, our whole discussion of radiation has been based on the formula for electric dipole transitions. However, we saw at the beginning of that discussion that electric dipole transitions provide just the first term in a systematic expansion of the electromagnetic transition amplitude. It is time to go back and study the higher terms in the expansion.

Return, then, to the equation that was the starting point for our derivation of radiation amplitudes. We wrote the Hamiltonian for a Schrödinger particle interacting with a classical electromagnetic wave as

$$\begin{aligned} \Delta H \;=\; & -\frac{q}{2m}\{\vec{p}, \vec{\epsilon} A_0\} \sin \omega t \\ & -\frac{q}{2m}\{\vec{p}, \vec{\epsilon} A_0 \vec{k}\cdot\vec{x}\} \cos \omega t \\ & - g \frac{q}{2m} \vec{k}\times\vec{\epsilon}\, A_0 \cdot \vec{S} \cos \omega t \end{aligned}$$

We transformed the first line into the electric dipole interaction. To do this, we took the matrix element between atomic states b and a and used the identity

$$\langle b| \frac{\vec{p}}{m} |a\rangle = -i \langle b| [\vec{r}, H_0] |a\rangle = +i(E_b - E_a) \langle b| \vec{r} |a\rangle$$

to write the matrix element of the first term as

$$\langle b| \Delta H |a\rangle = -iq\omega \langle b| \vec{\epsilon}\cdot\vec{r} |a\rangle A_0 \sin \omega t$$

where

$$\omega = E_b - E_a$$

We can simplify the next line of the formula in the same way. The second line involves the quantity

$$\frac{1}{2m} \left\{ \vec{p}, \; \vec{\epsilon} A_0 \; \vec{k} \cdot \vec{x} \right\}$$

Here $\vec{x}$ and $\vec{p}$ are the operators giving the position and momentum of the Schrödinger particle. We have to be careful to respect the fact that these operators do not commute. Write this quantity as

$$A_0 \frac{1}{2m} \epsilon^i k^j \quad (p^i x^j + x^j p^i)$$

and expand it as

$$\left(\frac{p^i}{2m} x^j + x^j \frac{p^i}{2m} \right)$$

$$= \frac{1}{4m} \left[(p^i x^j - p^j x^i) + (x^j p^i - x^i p^j) \right.$$
$$\left. + (p^i x^j + p^j x^i) + (x^j p^i + x^i p^j) \right]$$

$$= \frac{1}{4m} \left[\epsilon^{ijk} (\vec{p} \times \vec{x})^k + \epsilon^{jik} (\vec{x} \times \vec{p})^k \right]$$
$$+ \frac{1}{4m} \left[p^i x^j + p^j x^i + x^j p^i + x^i p^j \right]$$

In the last expression, the first line is

$$-\frac{1}{2m} \epsilon^{ijk} L^k \qquad\qquad \vec{L} = \vec{x} \times \vec{p}$$

The second line can be simplified by noting that

$$[x^i x^j, p^k] = i\,[x^i \delta^{jk} + x^j \delta^{ik}]$$

Then

$$-i\,[x^i x^j, \frac{p^2}{2m}] = -i\,[x^i x^j, p^k]\frac{p^k}{2m} - i\frac{p^k}{2m}[x^i x^j, p^k]$$

$$= \frac{-i}{2m}[x^i p^j + x^j p^i + p^i x^j + p^j x^i]$$

The second line of the Hamiltonian thus simplifies to

$$\frac{q}{2m}\,\vec{\epsilon}\times\vec{k}\cdot\vec{L}\quad A_0\cos\omega t$$

$$+\, i\frac{q}{2}\,\epsilon^i k^j\quad [x^i x^j, H_0]\; A_0\cos\omega t$$

We can add to this the third line of the Hamiltonian, in the form

$$\frac{q}{2m}\,\vec{\epsilon}\times\vec{k}\cdot g\vec{S}\quad A_0\cos\omega t$$

The final result is

$$\Delta H = iq\,[\vec{\epsilon}\cdot\vec{x}, H_0]\; A_0\sin\omega t$$
$$+\, i\frac{q}{2}\,\epsilon^i k^j\,[x^i x^j, H_0]\; A_0\cos\omega t$$
$$+\, \frac{q}{2m}\,\vec{\epsilon}\times\vec{k}\cdot(\vec{L}+g\vec{S})\; A_0\cos\omega t$$

Taking the matrix element between state b and a, this becomes

$$\langle b|\Delta H|a\rangle = -iq\omega \langle b|\vec{\varepsilon}\cdot\vec{x}|a\rangle A_0 \sin\omega t$$
$$-i\frac{q}{2}\omega \langle b|(\vec{\varepsilon})^i(\vec{k})^j x^i x^j|a\rangle A_0 \cos\omega t$$
$$+\frac{q}{2m}\vec{\varepsilon}\times\vec{k}\cdot\langle b|\vec{L}+g\vec{S}|a\rangle A_0\cos\omega t$$

It is now easy to related the various terms to specific operators that act on the atomic states. The first term of course has the electric dipole operator. In the second term, we can use the fact that $\vec{\epsilon}\cdot\vec{k}=0$ to rewrite the operator as

$$\varepsilon^i k^j \left(x^i x^j - \frac{1}{3}\delta^{ij} x^2\right)$$

This is the electric quadrupole operator. The third term contains the magnetic moment of the Schrödinger particle that we saw earlier in the interaction of a quantum particle with a constant magnetic field.

As we did earlier, we can relate the matrix elements given here to the matrix elements for absorption or emission of a single photon by squaring the various amplitudes, relating them to scattering rates, and then extracting the scattering rate for a single photon. This gives the same rule as that discussed in the previous lecture. We replace the factor of $\vec{A}$ by the factor

$$\left(\frac{1}{2\epsilon_0\omega}\right)^{1/2}$$

Now we have the matrix elements for absorption of a single photon by the three types of operators just discussed. In writing these expressions, I will assume that we will use these expressions to first order in perturbation theory, and so I will neglect relative phases. The matrix elements are then

$$\langle b|\Delta H|a\gamma\rangle = \begin{cases} q\left(\frac{\omega}{2\epsilon_0}\right)^{1/2} \langle b|\vec{\epsilon}\cdot\vec{x}\,|a\rangle & (E1) \\ q\left(\frac{\omega}{2\epsilon_0}\right)^{1/2}\cdot\frac{1}{2}\cdot\epsilon^i k^j \langle b|x^i x^j|a\rangle & (E2) \\ \frac{q}{2mc}\left(\frac{\omega}{2\epsilon_0}\right)^{1/2} \epsilon^{ijk}\,\epsilon^i \hat{k}^j \langle b|(\vec{L}+g\vec{S})^k|a\rangle & (M1) \end{cases}$$

These matrix elements mediate, respectively, *electric dipole (E1)*, *electric quadrupole (E2)*, and *magnetic dipole (M1)* transitions. The E2 amplitude is smaller than the E1 amplitude by a factor

$$k\cdot x$$

The M1 amplitude is smaller than the E1 amplitude by a factor

$$\frac{p}{mc}$$

Both of these quantities are of the order of

$$v_e/c \sim 10^{-2}$$

where v_e is the velocity of an electron in an atom. The next terms in the Taylor series expansion of ΔH give electric octopole (E3) and magnetic quadrupole (M2) transitions, which are suppressed by a further factor of v_e/c in the amplitude.

The various terms in ΔH have very different structures. We have already seen that the E1 amplitude gives zero rate for many transitions between states of the Hydrogen atom. For example, the E1 amplitude does not give transitions between different S states, or between D states and S states. The higher moments fill in some of these gaps by providing nonzero rates. So far, we have analyzed this anecdotally. However, there is a more systematic treatment.

The rules that govern which transitions are allowed and which are forbidden for each type of transition are called *selection rules.* We can derive general forms of the selection rules by using the symmetries of atomic physics. Some selection rules follow from angular momentum conservation. Additional rules follow from the application of discrete symmetries. I will now discuss these two types of rules in turn.

Consider first the rules following from angular momentum. To derive these, we note that the various multipole operators each have a definite spin, which can be added to the spin of the state a. The result must overlap with the spin of the state b. Consider first the dipole operator

$$\vec{\epsilon} \cdot \vec{x}$$

We can write this as

$$r \cdot \vec{\epsilon} \cdot (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

and then as a linear combination of wavefunctions with angular momentum $L = 1$.

$$\sqrt{\frac{4\pi}{3}}\ r\ \left[\left(\frac{\epsilon^1 + i\epsilon^2}{\sqrt{2}}\right) Y_{1-1}(\theta,\phi) + \left(\frac{\epsilon^1 - i\epsilon^2}{\sqrt{2}}\right) Y_{11}(\theta,\phi) + \epsilon^3\, Y_{10}(\theta,\phi)\right]$$

If the state a has angular momentum J, the state

$$\vec{\epsilon} \cdot \vec{x}\ |a\rangle$$

is then a linear combination of states with the angular momenta that appear when we add spins 1 and J_a,

$$1 \oplus J_a = J_a - 1 \,,\; J_a \,,\; J_a + 1$$

If the angular momentum of the state b is not one of these, the electric dipole matrix element will vanish. Then, an E1 transitions can only connect states with

$$|\Delta J| = 0, 1$$

Further, if the states have definite L, the same rule applies to L

$$|\Delta L| = 0, 1$$

This rule forbids electric dipole matrix elements linking

$$S \rightleftarrows D \qquad\qquad S, P \rightleftarrows F$$

The electric quadrupole operator has $L = 2$. It is a symmetric combination of two $L = 1$ operators, with the term proportional to δ^{ij} subtracted out. Then E2 transitions obey the selection rule

$$|\Delta J| = 0, 1, 2$$

This now allows

$$S \rightleftarrows D$$

but still forbids

$$S \rightleftarrows F$$

The magnetic dipole operator has $J = 1$, so it allows only

$$|\Delta J| = 0, 1$$

It does not change L, but it allows the spin configuration to change

$$(S = 0) \rightleftarrows (S = 1)$$

Then, for example, it can mediate transitions from the $J = 1$ to the $J = 0$ 1S states of Hydrogen.

A different argument for selection rules comes from application of discrete symmetries of the atomic Hamiltonian. I will discuss the most important of these symmetries, *parity*. In 3 dimensions, parity is best defined as reversal of all coordinates of space.

$$\begin{aligned} x^1 &\to -x^1 \\ x^2 &\to -x^2 \\ x^3 &\to -x^3 \end{aligned}$$

In quantum mechanics, this operation is implemented by a unitary operator P that acts on states according to

$$P\,\psi(\vec{x},t) = \psi(-\vec{x},t) \qquad P^2 = 1$$

and acts on operators according to

$$\mathcal{P}\vec{x}\mathcal{P} = -\vec{x} \qquad \mathcal{P}\vec{p}\mathcal{P} = -\vec{p}$$

Vector operators are then ordinarily *odd* under parity. But notice that

$$\mathcal{P}\ \vec{x}\times\vec{p}\ \mathcal{P} = (-\vec{x})\times(-\vec{p}) = +\vec{x}\times\vec{p}$$

so angular momentum – and, in general, any vector defined as a cross product – will be *even* under parity. At this moment, it is unclear how the internal spin of a particle transforms under parity. It is most useful to assign the transformation property

$$\mathcal{P}\ \vec{S}\ \mathcal{P} = +\vec{S}$$

The definition of parity requires a little more explanation. Parity looks like a rotation in 3 dimensional space. A very similar operation

$$\mathcal{R}(\pi\hat{3}):\quad \begin{array}{l} x^1 \to -x^1 \\ x^2 \to -x^2 \\ x^3 \to +x^3 \end{array}$$

is a 180° rotation about the $\hat{3}$ axis. However, there is a difference between this transformation and parity. A rotation is sometimes defined as a linear transformation on space that preserves the inner product $\vec{x}\cdot\vec{y}$. A rotation is implemented by a 3×3 matrix R

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \to R\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}.$$

The condition that R preserves inner products implies

$$\det R = \pm 1$$

But, the determinant of R is a continuous function of the rotation angles. This means that any rotation that is generated continuously from the identity transformation $R = 1$ must have

$$\det R = 1$$

It is easy to write down matrices R that have determinant equal to (-1), for example,

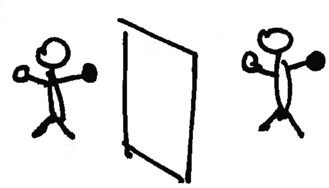
$$R = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \quad \text{or} \quad R = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

However, these transformations cannot be continuously from the identity; they require a discrete operation in addition. It is clearer to define *rotations* R to be operations for which $\det R = 1$. Then the operations for which the determinant is (-1) are written

$$R = R_{(p)} \cdot P \qquad\qquad \det R_{(p)} = +1$$

where P is parity. For example, mirror reflection in the $\hat{3}$ axis

$$M: \quad \begin{aligned} x^1 &\to x^1 \\ x^2 &\to x^2 \\ x^3 &\to -x^3 \end{aligned}$$

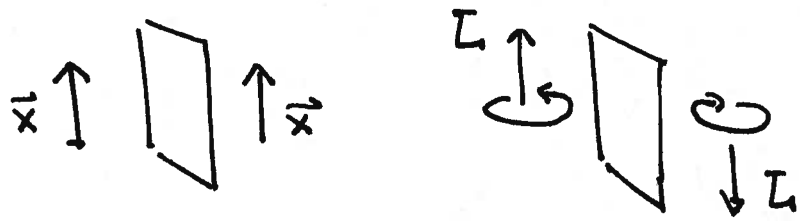


is written

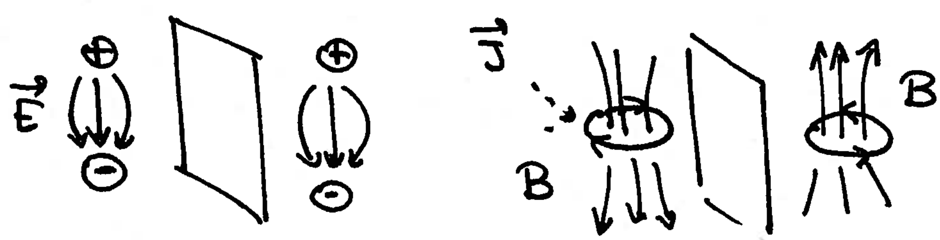
$$M = R(\pi\hat{z}) \cdot P$$

Note that in 2 dimensions, the definition of parity that I have given above is actually a rotation; however, mirror reflection is an operation distinct from rotations in any dimension.

It is often useful to think about parity through analysis of mirror reflection. For example, this makes it clear in a picture that $\vec{x}$ and $\vec{L}$ have opposite transformation properties



Similarly, a constant $\vec{A}$ or $\vec{E}$ field will be *odd* under parity, while a constant $\vec{B}$ field, built from the cross product, will be *even*. The picture using mirror-reflection is



Since a rotation can be implemented in quantum mechanics as

$$R = e^{-i\vec{\alpha}\cdot\vec{J}}$$

all rotations, as defined above, will be symmetries of the quantum mechanical problem if $\vec{J}$ commutes with the Hamiltonian. However, this does *not* imply that P is a symmetry. It must be separately checked whether P commutes with the Hamiltonian or not. For the interaction of electromagnetic fields with a Schrödinger particle, the Hamiltonian is

$$\mathcal{H} = \frac{(\vec{p} - q\vec{A})^2}{2m} + q\Phi$$

If we assign the external fields the transformations under parity

$$P\,\Phi(\vec{x})\,P = \Phi(-\vec{x}) \qquad P\,\vec{A}(\vec{x})\,P = -\vec{A}(-\vec{x})$$

which will actually be realized when we quantize these fields, then this Hamiltonian commutes with P. In a similar way, the Hamiltonian describing the strong interactions of protons and neutrons commutes with parity. However, the Hamiltonian that describes the weak interactions responsible for radioactive decay processes does not commute with P. Indeed, at a truly fundamental level, the laws of physics are not parity-invariant, and the parity-invariance of electrodynamics looks like an accident. However, electromagnetism dominates the structure of atoms and so, in atomic physics, the consequences of parity invariance apply and are very useful.

If

$$[H, P] = 0$$

then H and P can be simultaneously diagonalized. This means that we can find a basis in which all states $|\psi\rangle$ are eigenstates of H and also obey

$$P\,|\psi\rangle = P_\psi\,|\psi\rangle$$

Since $P^2 = 1$,

$$\mathcal{P}_\psi = +1 \quad \text{or} \quad \mathcal{P}_\psi = -1$$

We call these states of *even* or *odd parity*, respectively.

With this framework, we can work out the implications of parity for photon transitions. Begin by analyzing the electric dipole matrix element between the states b and a. Since $P^2 = 1$, we can manipulate this as follows:

$$\begin{aligned}
\langle b|\vec{\varepsilon}\cdot\vec{x}|a\rangle &= \langle b|\mathcal{P}\mathcal{P}\, x^i\, \mathcal{P}\mathcal{P}|a\rangle\, \varepsilon^i \\
&= \mathcal{P}_b \langle b|(-x^i)|a\rangle\, \mathcal{P}_a\, \varepsilon^i \\
&= (-\mathcal{P}_b\mathcal{P}_a)\, \langle b|\vec{\varepsilon}\cdot\vec{x}|a\rangle
\end{aligned}$$

In the last term, we come back to the original amplitude, times a factor that can be $+1$ or -1. If the factor is -1, the amplitude is equal to its negative, so it must be zero. We conclude that E1 transitions link states of *opposite* parity. This criterion already forbids E1 transitions

$$S \rightleftarrows D$$

without the need for arguments involving angular momentum. However, it is not strong enough to forbid transitions

$$S \rightleftarrows F$$

Using the same method, we can read off the parity selection rules for E2 and M1 transitions. The E2 operator

$$\vec{\varepsilon}\cdot\vec{\underset{\sim}{x}}\quad \vec{L}\cdot\vec{\underset{\sim}{x}}$$

is even under parity and thus links states of the same parity. The M1 operator

$$\vec{L}+g\vec{S}$$

is also even under parity and thus also connects states of the same parity. Notice that the hyperfine levels of Hydrogen

$$|1S\quad S=0\rangle \qquad\qquad |1S\quad S=1\rangle$$

have the same parity, since they differ only in their spin configuration. All of our selection rules also allow the M1 transition

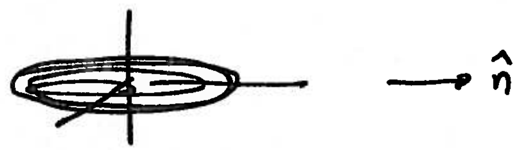
$$|1S\quad S=0\rangle \qquad\qquad |2S\quad S=1\rangle$$

But still the transition amplitude is very small. Since the M1 operator does not act on the spatial wavefunction, the transition amplitude computed in lowest order involves the overlap

$$\langle 1S|2S\rangle$$

which is zero. If we take into account the recoil of the Hydrogen atom and other relativistic effects, there is a small but nonzero overlap. A transition with this property is called a *forbidden* M1 transition.

I would like to illustrate the E2 transition amplitude with an interesting application in nuclear physics. In the rare earth regions of the periodic table, the balance of forces between short-ranged attraction of nucleons and the electrostatic repulsion of the protons causes nuclei to deform to the shape of a prolate spheroid



At this level of approximation, the nucleus has a definite orientation, given by the direction $\hat{n}$ of its long axis. However, a quantum mechanical system cannot stay in a fixed orientation. Rather, its orientation fluctuates, and the ground state will be a coherent superposition of the various orientations. We can describe this by a Hamiltonian

$$\mathcal{H} = \frac{L^2}{2I}$$

where I is the moment of inertia of the nucleus about its short axes. The eigenstates of this Hamiltonian are

$$Y_{\ell m}(\hat{n}) \quad \text{with} \quad E = \frac{\ell(\ell+1)}{2I}\hbar^2$$

Remember that

$$Y_{\ell m}(-\hat{n}) = (-1)^\ell \, Y_{\ell m}(\hat{n})$$

But, the same nucleus is described equally well by the axis $\hat{n}$ or $-\hat{n}$. This means that the states of odd ℓ actually do not exist. The low-energy states of the nucleus then form a *rotational band* with energies

$$E = \frac{\hbar^2\,\ell(\ell+1)}{2I} \qquad \ell = 0, 2, 4, 6 \ldots$$

All of these states have parity $P = +1$.

The states of the rotational band can be excited by bombarding a rare earth nucleus with α particles. An α particle hitting the nucleus at high impact parameter sets the nucleus in rotational motion. The rotating state then decays to the rotational ground state.

An example of such a spectrum, the rotational band in ^{160}Dy, is shown in the figure. The top figure, from Johnson, Ryde, and Hjoth, Nucl. Phys. A179, 753 (1972), shows the photons from the de-excitation cascade, which I will now discuss.

The transitions between states of the rotational band cannot be mediated by E1 transitions, since these states have the same parity and are spaced by $\Delta L = 2$. An E2 transition can take the state only to the next lower rotational state. Thus, the decay process emits a cascade of gamma rays, each one lowering the energy by one step in the band.

I would like to compute the rate of one of these decays. Using Fermi's Golden Rule, the decay rate is

$$\Gamma = \int d\Pi \; \left| \langle (\ell-2)\,\gamma \,|\, \Delta H \,|\, \ell \rangle \right|^2$$

Putting in the electric quadrupole matrix element for ΔH, we have

$$\Gamma = \frac{\omega^2}{\pi c^3} \cdot \frac{Q^2}{4} \cdot 2\pi\alpha c\,\omega \cdot \int \frac{d\Omega}{4\pi} \left| \langle \ell-2 |\, \vec{\epsilon}^{\,*}\cdot\vec{x} \;\hat{k}\cdot\vec{x} \,| \ell \rangle \right|^2$$

$$= \frac{\pi}{2}\alpha\, Q^2 R^4 \,\frac{\omega^5}{c^4} \int \frac{d\Omega}{4\pi} \left| \langle \ell-2 |\, \vec{\epsilon}^{\,*}\cdot\hat{x} \;\hat{k}\cdot\hat{x} \,| \ell \rangle \right|^2$$

where Q is the total charge of the rotating clump of nucleons and R is its size. These quantities are related to the moment of inertia I by

Johnson, Ryde, & Hjorth Nucl. Phys. A179, 753 (1972)

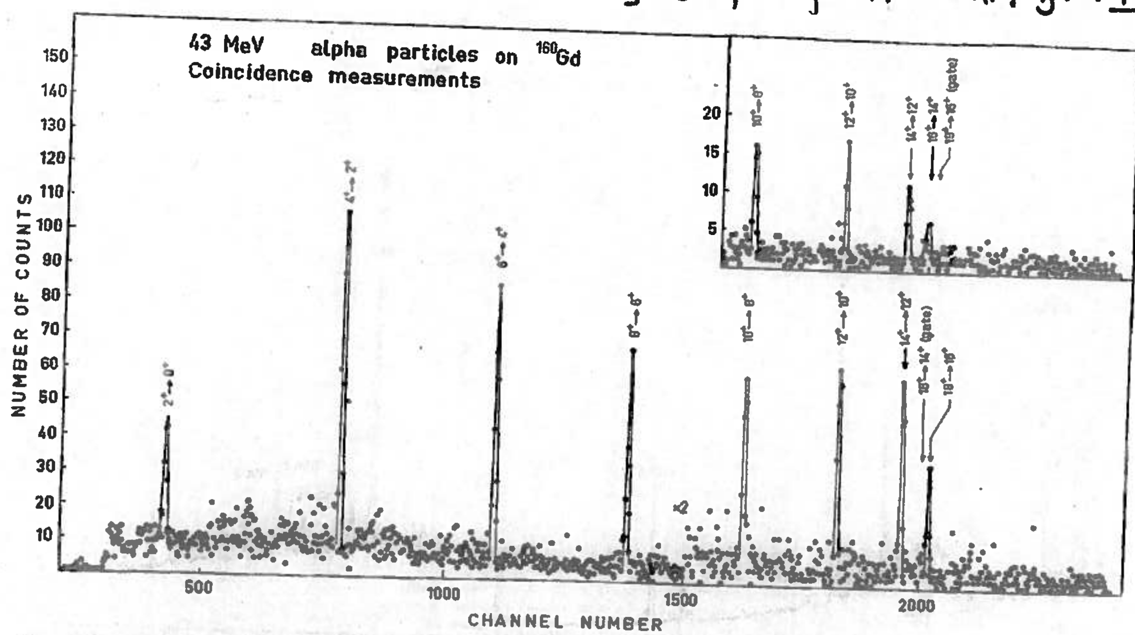
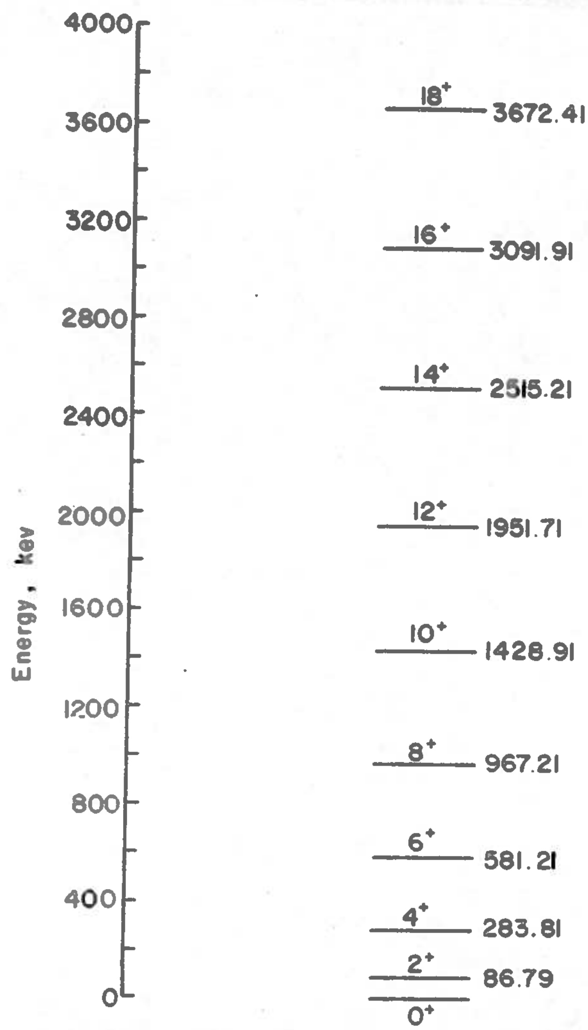


Fig. 5. Coincidence spectra showing the cascade deexciting the ground state band in ^{160}Dy. The gates are set on the 576.7 and 580.5 keV peaks. The background is not subtracted in this case.



spectrum of ^{160}Dy from Preston & Bhaduri Structure of the Nucleus based on this data

$$QR^2 = \frac{1}{2}\frac{I}{m_p}$$

Using this relation, the decay rate becomes

$$\Gamma = \frac{\pi\alpha}{8\pi}\left(\frac{\omega}{c}\right)^5 \cdot c\left(\frac{I}{m_p}\right)^2 \cdot \int\frac{d\Omega}{4\pi}\, |a|^2$$

with

$$a = \int d\Omega\; Y^*_{\ell-2\, m'}(\theta,\phi)\;(\vec{\varepsilon}^{\,*}\cdot\hat{x})(\hat{k}\cdot\hat{x})\; Y_{\ell m}(\theta,\phi)$$

I will evaluation this integral in one special case. For a rotational state with $L = \ell$, the highest m value is $m = \ell$. If this state makes a transition to the $L = (\ell - 2)$ state, on the highest m state of that level, $m = (\ell - 2)$, can be accessed by the E2 matrix element. Then the only nonzero amplitude is

$$m' = \ell - 2$$

To evaluate this, we need

$$Y_{\ell\ell}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}\;\frac{[(2\ell+1)!]^{1/2}}{2^\ell\,\ell!}\;\sin^\ell\theta\; e^{i\ell\phi}$$

The normalization reflects

$$\int_{-1}^{1} d\cos\theta\;\sin^{2\ell}\theta = 2\cdot\frac{2^\ell\ell!}{1\cdot3\cdot5\cdots(2\ell+1)} = 2\cdot\frac{2^{2\ell}(\ell!)^2}{(2\ell+1)!}$$

Only the part of the operator

$$\vec{\epsilon}^{\,*} \cdot \hat{x} \quad \hat{k} \cdot \hat{x}$$

with ϕ-dependence $e^{-2i\phi}$ can contribute. If we write

$$\hat{x} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

we have

$$\vec{\epsilon}^{\,*} \cdot \hat{x} = \frac{(\epsilon^*)^1 + i(\epsilon^*)^2}{\sqrt{2}} \frac{\sin\theta\, e^{-i\phi}}{\sqrt{2}} + \frac{(\epsilon^*)^1 - i(\epsilon^*)^2}{\sqrt{2}} \frac{\sin\theta\, e^{i\phi}}{\sqrt{2}}$$

$$+ \epsilon^3 \cos\theta$$

and so

$$\vec{\epsilon}^{\,*} \cdot \hat{x} \; \hat{k} \cdot \hat{x} = \frac{\epsilon^{*1} + i\epsilon^{*2}}{\sqrt{2}} \frac{\hat{k}^1 + i\hat{k}^2}{\sqrt{2}} \frac{\sin^2\theta}{2} e^{-2i\phi} + \cdots$$

This expression is nonzero only for the choice

$$\vec{\epsilon} = \frac{\hat{1} + i\hat{2}}{\sqrt{2}} = \vec{\epsilon}_+ \quad \epsilon^* = \frac{\hat{1} - i\hat{2}}{\sqrt{2}} = \vec{\epsilon}_- \quad \text{so} \quad \frac{(\epsilon^*)^1 + i(\epsilon^*)^2}{\sqrt{2}} = 1$$

Then the spin angular momentum of the photon is $S^3 = +1$, and the photon also carries away orbital angular momentum $L^3 = +1$. In this way, the angular momentum of the rotating nucleus is released to the emitted photon. The amplitude becomes

$$a = \int d\Omega \, \frac{1}{4\pi} \frac{[(2\ell+1)!(2\ell-3)!]^{\frac{1}{2}}}{2^{\ell}\ell! \, 2^{\ell-2}(\ell-2)!} (\vec{\epsilon}^{\,*}\cdot\vec{\epsilon}_+)\left(\frac{\hat{k}^1+i\hat{k}^2}{\sqrt{2}}\right) \frac{\sin^{2\ell}\theta}{2}$$

$$= \frac{[(2\ell+1)!(2\ell-3)!]^{\frac{1}{2}}}{2^{\ell}\ell! \, 2^{\ell-2}(\ell-2)!} \cdot \frac{1}{2} \cdot \frac{2^{2\ell}(\ell!)^2}{(2\ell+1)!} (\vec{\epsilon}^{\,*}\cdot\vec{\epsilon}_+)\left(\frac{\hat{k}^1+i\hat{k}^2}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left[\frac{2\ell(2\ell-2)}{(2\ell+1)(2\ell-1)}\right]^{\frac{1}{2}} (\vec{\epsilon}^{\,*}\cdot\vec{\epsilon}_+)\left(\frac{\hat{k}^1+i\hat{k}^2}{\sqrt{2}}\right)$$

The energy of the photon in the transition is

$$\omega = ck = \frac{\ell(\ell+1)}{2I} - \frac{(\ell-2)(\ell-1)}{2I} = \frac{4\ell-3}{2I}$$

Assembling this factor and the value of the amplitude, we find

$$\Gamma = \frac{\alpha}{8}\left(\frac{4\ell-3}{2Ic}\right)^5 \cdot c \cdot \left(\frac{I}{m_p}\right)^2 \cdot \int \frac{d\Omega}{4\pi} \frac{1}{4}\left[\frac{2\ell(2\ell-2)}{(2\ell+1)(2\ell-1)}\right] \frac{\sin^2\theta}{2} \sum_{\epsilon} |\vec{\epsilon}^{\,*}\cdot\vec{\epsilon}_+|^2$$

$$= \frac{\alpha}{256}(4\ell-3)^2\left(\frac{4\ell-3}{2I}\right)^3 \frac{1}{[m_pc^2]^2}\left[\frac{(2\ell)(2\ell-2)}{(2\ell+1)(2\ell-1)}\right] \int \frac{d\Omega}{4\pi} \sin^2\theta \sum_{\epsilon} |\vec{\epsilon}^{\,*}\cdot\vec{\epsilon}_+|^2$$

The polarization sum in this expression is

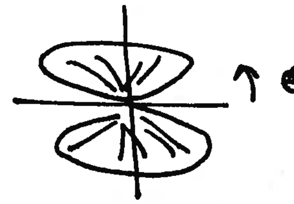
$$\sum_{\epsilon} |\vec{\epsilon}^{\,*}\cdot\vec{\epsilon}_+|^2 = \epsilon_-^i(\delta^{ij} - \hat{k}^i\hat{k}^j)\epsilon_+^j = \left(1 - \frac{1}{2}\sin^2\theta\right) = \frac{1}{2}(1+\cos^2\theta)$$

so that the integral over the photon angle is

$$\int \frac{d\Omega}{4\pi} \sin^2\theta \, \frac{1+\cos^2\theta}{2} = \int \frac{d\Omega}{4\pi} \frac{1-\cos^4\theta}{2} = \frac{2}{5}$$

We now have the complete expression for the decay rate. The angular distribution is

$$\frac{d\Gamma}{d\cos\theta} \sim (1 - \cos^4\theta)$$



and the total rate is

$$\Gamma = \frac{\alpha}{40}\left(\ell - \frac{3}{4}\right)^2 \frac{2\ell(2\ell-2)}{(2\ell+1)(2\ell-1)} \cdot \left(\frac{4\ell-3}{2I}\right)^3 \hbar^6 \frac{1}{[m_p c^2]^2}$$

For large ℓ this becomes

$$\Gamma \sim \frac{\alpha}{40} \ell^2 (\hbar\omega)^3 \frac{1}{[m_p c^2]^2}$$

This expression increases as ℓ becomes large, but it starts with a small coefficient. Thus

$$\frac{\Gamma}{\hbar\omega} \sim \frac{\alpha}{40} \ell^2 \left(\frac{\hbar\omega}{m_p c^2}\right)^2$$

This means that states up to $\ell \sim 20$ are in principle visible as distinct gamma ray lines. By this criterion, we should be able to see up to 10 E2 photons per event. In practice, the upper limit on ℓ is set by the breakdown of the simple rigid body model we are using here at large values of ℓ.