May 29

Positronium and Quarkonium

A wonderful example of the role of selection rules and discrete symmetries is given by the Hydrogenic atoms consisting of a spin $\frac{1}{2}$ particle and its antiparticle. The canonical example is *positronium*, the bound state of an electron and a positron. To produce positronium, positrons created in radioactive decay (β^+ emission) move through a gas and lose energy, eventually coming down to a kinetic energy of a few eV. At this point, the positrons can capture atomic electrons and form positronium. As I will discuss, the lifetime of positronium is long enough to allow the observation of many features of its atomic spectrum.

There are three new features that arise in treating positronium rather that Hydrogen. The first is of course the fact that positronium can annihilate. The second is that, to reduce the problem of two equal-mass particles to a simple bound state problem with one coordinate, the particle mass in the bound state equations must be taken to be the *reduced mass*

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{1}{2} m_e$$

The the Bohr radius and the 1S binding energy in positronium are given by

$$a_0^{(pos)} = \frac{1}{\alpha}\frac{\hbar}{\mu c} = 2a_0 \qquad E_b(1S) = \frac{1}{2}\frac{\alpha}{a_0^{pos}} = \frac{1}{2} Ry$$

The third feature is less trivial. The electron and the positron are spin $\frac{1}{2}$ particles with the same mass and exactly opposite electric charges. So, the exchange of electrons and positrons should be a symmetry. To be more precise, we define the following operation, called *charge conjugation*:

$$C\,|e^-(\vec{p},s)\rangle = |e^+(\vec{p},s)\rangle$$
$$C\,|e^+(\vec{p},s)\rangle = |e^-(\vec{p},s)\rangle$$

This satisfies $C^2 = 1$. Charge conjugation reverses the signs of electric charges; to compensate this, we must have the action

$$C \, \vec{A}(\vec{x},t) \, C = -\vec{A}(\vec{x},t)$$

so that

$$C \, |\gamma(\vec{p},\vec{\epsilon})\rangle = -|\gamma(\vec{p},\vec{\epsilon})\rangle$$

The operation C commutes with the Hamiltonian that couples electrons and positrons to the electromagnetic field. It can be shown that C commutes with the complete Hamiltonian of quantum electrodynamics.

Since C is a discrete symmetry, we need to investigate its validity for each of the fundamental interactions. It turns out that C is also a symmetry of the strong nuclear interaction, and that, like P, it is not a symmetry of the weak interactions. The combined operation CP commutes with almost all of the terms in the Hamiltoian of the weak interactions, but it is violated by specific weak interactions involving heavy quarks.

Electromagnetism determines the structure of positronium, and so we can assign definite C and P quantum numbers to the various states. In assigning P, there is one additional fact that we need to know. A quantum state can obtain a parity from the form of its wavefunction, but, also, a particle can have an *intrinsic parity.* The action of P on the quantum state of a particle has the form

$$P \, |\phi(\vec{p},s)\rangle = \eta_\phi \, |\phi(-\vec{p},s)\rangle$$

but the factor $\eta(\phi)$ can be either $+1$ or -1, depending on the identity of the particle. Dirac's relativistic theory of spin $\frac{1}{2}$ particles implies that each such particle has an antiparticle with the opposite electric charge. As a corollary, it implies that these two particles have opposite intrinsic parity

$$\eta_{e^-} = -\eta_{e^+}$$

We can choose the convention that the intrinsic parity of the electron is (+1); then the intrinsic parity of the positron must be −1. Since

$$\mathcal{P} : \vec{A}(\vec{x}) \longrightarrow -\vec{A}(-\vec{x})$$

the photon has intrinsic parity −1.

We can now assign to the various states of positronium values for their P and C quantum numbers. P is easier. The spins of the e^+ and e^- are left unchanged by P. Then the states with orbital angular momentum L have

$$P = (-1)(-1)^L = (-1)^{L+1}$$

The extra factor of −1 is the intrinsic parity of the e^+.

Next, compute C. For states of definite momentum

$$C \,|\, e^-(\vec{p}, s)\, e^+(\vec{q}, s') \rangle = |\, e^+(\vec{p}, s)\, e^-(\vec{q}, s') \rangle$$

The electron and positron are fermions. So, restoring their original positions, which requires an interchange of particles, gives a minus sign

$$= -\,|\, e^-(\vec{q}, s')\, e^+(\vec{p}, s) \rangle$$

We also need to interchange $\vec{p}$ and $\vec{q}$. This depends on the symmetry of the orbital wavefunction and gives a factor $(-1)^L$. Finally, there is another possible factor from the spin configuration. The possible spin states are

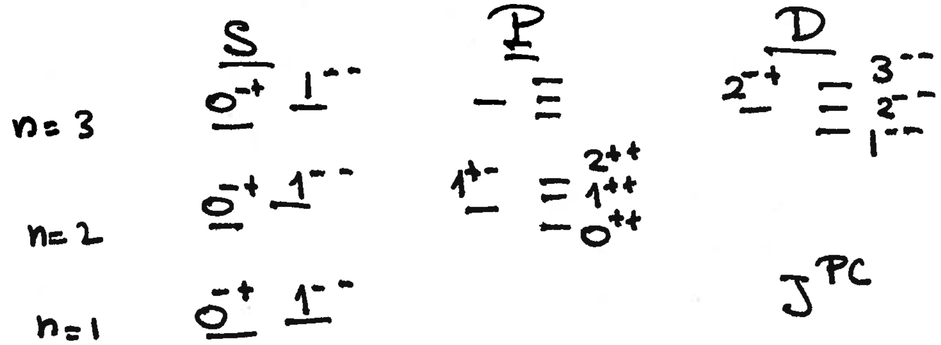
$$S = 0 \qquad \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$S = 1 \qquad |\uparrow\uparrow\rangle \qquad \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \qquad |\downarrow\downarrow\rangle$$

When the e^+ and e^- are interchanged, their spins are interchanged. This gives a factor -1 for $S = 0$, $+1$ for $S = 1$. In all, we find for positronium bound states

$$C = (-1)^{L+S}$$

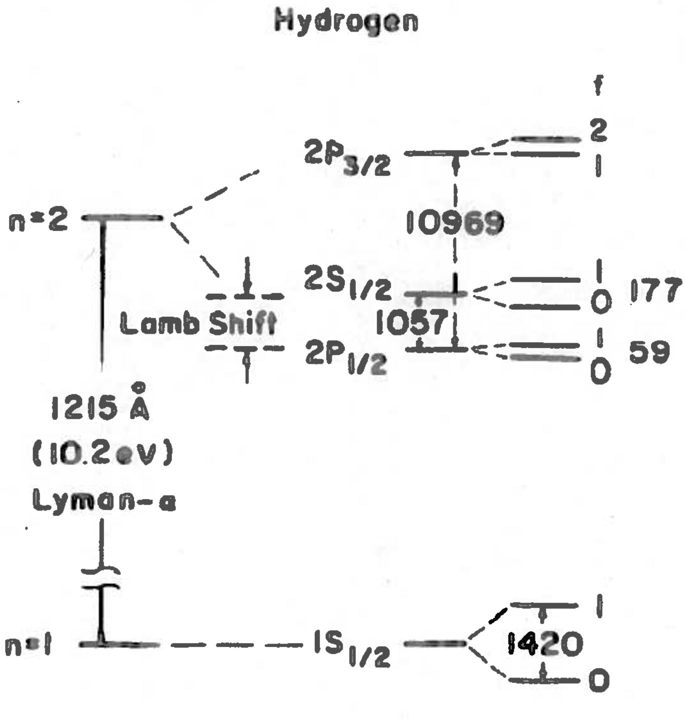
Here is a picture of the spectrum:

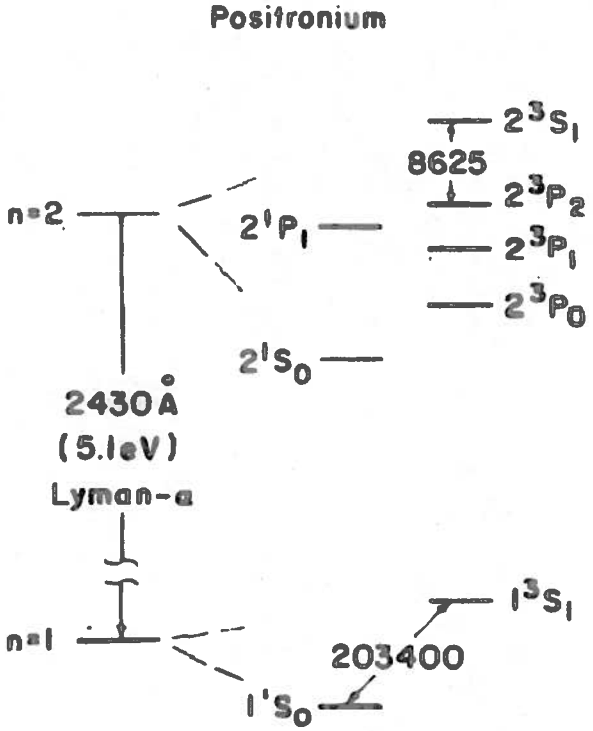


The measured energy level spectra of Hydrogen and positronium are compared in the figure, taken from Berko and Pendleton, Ann. Rev. Nucl. Part. Sci. 30, 543 (1980).

The C assignments have two important implications. First, the photon has $C = -1$, and so photon transitions change the value of C. Then, many expected transitions are forbidden. For example, the $J = 0$ 3S state cannot decay to any of the $S = 1$ 2P states – at any level of the multipole expansion – because such a transition would violate C.

Second, the C quantum number affects annihilation of the electron and positron. In order for the e^+ and e^- to annihilate, they must come to the same point in space, within an electron Compton wavelength


Hydrogen
f
2
1
$2P_{3/2}$
n=2
10969
$2S_{1/2}$
1
0
177
Lamb Shift
1057
$2P_{1/2}$
1
0
59
1215 Å
(10.2 eV)
Lyman-α
n=1
$1S_{1/2}$
1
1420
0


Positronium
2^3S_1
8625
2^3P_2
n=2
2^1P_1
2^3P_1
2^3P_0
2^1S_0
2430 Å
(5.1 eV)
Lyman-α
n=1
1^3S_1
203400
1^1S_0

$$\frac{\hbar}{m_e c} = \alpha \cdot a_0$$

So annihilation is highly suppressed except for S states. Other states of positronium typically, then, emit photons to radiate down to S states. The rates for E1 transitions are of the order of

$$\Gamma \sim \alpha^3 R_y$$

as we have seen for Hydrogen. The annihilation rate of the S states depends crucially on their spin. An e^+e^- state cannot decay to 1 photon; this violates momentum conservation. A 2-photon state has $C = +1$. Thus, states with $L = 0$, $S = 0$ can decay to 2 photons, but states with $L = 0$, $S = 1$ cannot. The rate for decay to 3 photons is suppressed by an additional factor of α. Specifically, for the 1S state, the $J = 0$ state, called *parapositronium*, has a decay rate to 2 photons

$$\Gamma = \frac{1}{2}\alpha^5 mc^2 = \alpha^3 R_y \qquad \tau = 1.2 \times 10^{-10}\,\text{sec}$$

The $J = 1$ state, called *ortho-positronium*, has a decay rate to 3 photons given by

$$\Gamma = \frac{2}{9\pi}(\pi^2 - 9)\,\alpha^6 mc^2 = 0.12\,\alpha^4 R_y \qquad \tau = 1.4 \times 10^{-7}\,\text{sec}$$

Then, when we shoot positrons into a gas and let them form positronium, 25% annihilate rapidly, in a tenth of a nanosecond, while the other 75% take 1000 times longer to annihilate.

If we form the 3S $J = 1$ state of positronium, this will have plenty of time to decay to the 2P states with $C = +1$. It is interesting to work out the relative decay rates. The E1 transitions are mediated by the operator

$$\left(\frac{\omega}{2\epsilon_0}\right)^{1/2} \vec{\epsilon}\cdot\left(e\,\vec{r}_+ + (-e)\,\vec{r}_-\right)$$

which is just

$$e\left(\frac{\omega}{2\epsilon_0}\right)^{1/2} \vec{\epsilon}\cdot\vec{r}$$

where $\vec{r} = \vec{r}_+ - \vec{r}_-$ is the relative coordinate in the bound state. The matrix elements of this operator are built from the amplitudes

$$\langle 2P\ m'\ S=1\ S^{3\prime} \,|\, r^m \,|\, 3S\ \ S=1\ \ S^3 \rangle$$

where $m, S^3 = -1, 0, 1$. The values of S and S^3 are not changed by the E1 matrix element. This amplitude has the simple structure

$$\langle m'\ S^{3\prime} \,|\, r^m \,|\, S^3 \rangle = a \cdot \delta^{mm'} \delta^{S^3 S^{3\prime}}$$

Now consider, for definiteness, the decays of the state with $S^3 = +1$. The E1 transition can change the value of m by at most 1 unit. Then the possible final states are

$$|J=2\ J^3=2\rangle = |m=1\ S^3=1\rangle$$

$$|J=2\ J^3=1\rangle = \frac{1}{\sqrt{2}}\left(|m=0\ S^3=1\rangle + |m=1\ S^3=0\rangle\right)$$

$$|J=1\ J^3=1\rangle = \frac{1}{\sqrt{2}}\left(|m=0\ S^3=1\rangle - |m=1\ S^3=0\rangle\right)$$

$$|J=2\ J^3=0\rangle = \frac{1}{\sqrt{6}}\left(|m=-1\ S^3=1\rangle + 2\,|m=0\ S^3=0\rangle + |m=1\ S^3=-1\rangle \right)$$

$$|J=1\ J^3=0\rangle = \frac{1}{\sqrt{2}}\left(|m=-1\ S^3=1\rangle - |m=1\ S^3=-1\rangle \right)$$

$$|J=0\ J^3=0\rangle = \frac{1}{\sqrt{3}}\left(|m=-1\ S^3=1\rangle - |m=0\ S^3=0\rangle + |m=1\ S^3=-1\rangle \right)$$

We can read off from these expressions the relative rates

$$\begin{array}{llllllll} 3S\ S=1\ S^3=1 \longrightarrow & 1 & : \frac{1}{2} & : \frac{1}{2} & : \frac{1}{6} & : \frac{1}{2} & : \frac{1}{3} \\ \quad J\ J^3 & (2,2) & (2,1) & (1,1) & (2,0) & (1,0) & (0,0) \end{array}$$

Then the decay widths to the states with $J = 0, 1, 2$ are in the ratio

$$\frac{1}{3} : 1 : \frac{5}{3} \qquad \text{or} \qquad 1 : 3 : 5$$

which is just the ratio of the number of states $(2J + 1)$.

The 3S 1^{--} state can also decay to the 3S 0^{-+} state by an M1 transition. However, this decay is highly suppressed relative to the decays to the 2P by the factor

$$\Gamma \sim \omega^3$$

where ω for this decay is a splitting of the fine-structure size

$$\Delta E \sim \alpha^2 R_y$$

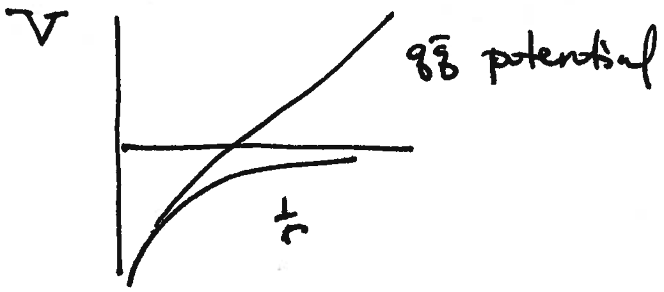
The M1 decays to lower S states are forbidden transitions, suppressed by the orthogonality of the spatial wavefunctions.

Many properties of positronium have been measured to very high precision as tests of quantum electrodynamics. For example, Steven Chu, Alan Mills, and John Hall measured the Rydberg energy in positronium to 12 parts per billion (Phys. Rev. Lett. 52, 1689 (1984))

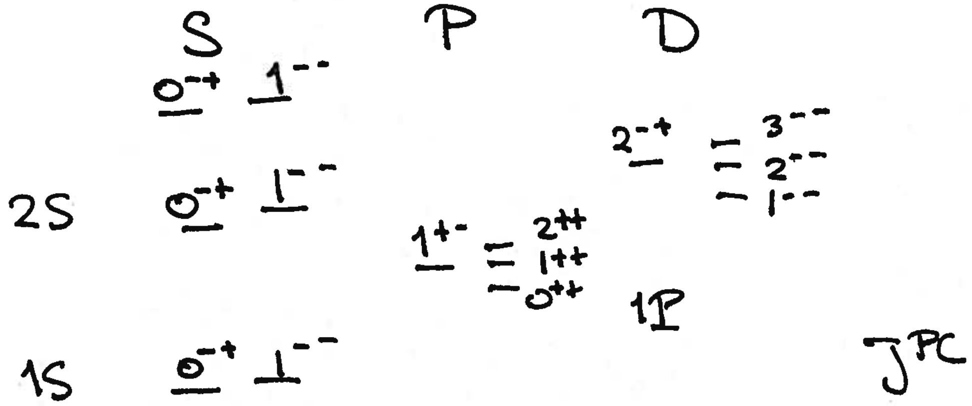
$$\frac{\Delta E}{\hbar}\left(2S(1^{--}) - 1S(1^{--})\right) = 1\;233\;607\;185 \pm 15 \text{ MHz}$$

This elegant measurement, done at Bell Labs before Chu came to Stanford, used two oppositely directed lasers to induce a 2-photon absorption process linking the two $C = -1$ states. The 2-photon technique has the advantage that the Doppler shifts from the thermal motion of the positronium atoms in the gas cancel between the two laser beams.

Strongly interacting particles like protons and neutrons, collectively called *hadrons*, are supposed to be made out of *quarks*, which are postulated to be elementary spin $\frac{1}{2}$ particles If this is correct, there ought to be *quarkonium* atoms, bound states of a quark and its antiparticle. Quarks are not seen in isolation; they are only found inside hadrons. This means that the potential between quarks and antiquarks cannot be a Coulomb potential but, instead, must increase at large distances



This has the effect that it raises the energies of states that lie at larger distances from $\vec{r} = 0$. Starting from the spectrum of Hydrogen, the change to a potential of this form moves the 2S up above the 2P, the 3S above the 3P above the 3D, etc. In quarkonium, the 2P is usually relabeled as the 1P, the 3D as the 1D, and so forth. The form of the expected spectrum is then



Quarkonium states can be make by recombination of the many quarks and antiquarks produced in high energy hadron collisions. They can also be made in a very simple way using electron-positron annihilation at high energy, through the elementary process

$$e^+e^- \longrightarrow q\bar{q}$$

The intermediate state is one quantum of electromagnetic radiation, a *virtual photon.* Then, the state that is created must be an S state with $C = -1$. The successive S states appear as prominent resonances in the e^+e^- annihilation cross section. These states can annihilate back to e^+e^- or multi-hadron final states, or they can decay to lower states in the quarkonium spectrum.

The strong interactions are mediated by a spin 1 particle similar to the photon, called the *gluon.* The same argument from C used above implies that the $C = -1$ quarkonium states have a highly suppressed decay to 3 gluons and are therefore should be very long-lived or appear as very narrow resonances. This is actually correct, but historically the discovery of very narrow quarkonium resonances was a tremendous surprise to particle physicists. The quarkonium spectrum of states is especially clear for the bound states of the heavy quarks c (charm) and b (bottom) and their antiparticles. The 1S 1^{--} state of $c\bar{c}$ has a mass of 3.1 GeV and a width of 93 keV. Its discovery caused a sensation, since typical hadrons of this mass have widths of many hundred of MeV. The discovery was made simultaneously at Brookhaven, using proton-proton collisions and observing the decay to $\mu^+\mu^-$, and at SLAC here at Stanford, using colliding beams of e^+ and e^-. The results were announced on the same day, November 12, 1974, and are published in Phys. Rev. Lett 33, 1404 and

1406 (1974). These particle is still called the J/ψ, with the names given by the two competing groups.

Subsequent measurements in e^+e^- annihilation at SLAC and elsewhere established the full charmonium spectrum. The current status is shown in the figure, taken from Eichten, Godfrey, Mahlke, and Rosner, Rev. Mod. Phys. 80, 1161 (2008). The E1 transitions

$$2S(1^{--}) \to 1P(0,1,2^{++}) \to 1S(1^{--})$$

are very prominent. The 1:3:5 rule for the E1 rates, modified by the ω^3 factor, which is a significant effect, is verified for the 2S to 1P transitions. Much of the structure of these radiative transitions is seen in the beautiful figure, from the Crystal Ball experiment at SLAC, shows the photon energy spectrum observed at the $\psi(2S)$ resonance. The E1 transitions are clearly seen, and there are smaller peaks showing the M1 transitions to the 2S and 1S 0^{-+} states. The figure is taken from Elliott Bloom's SLAC Summer Institute lecture notes SLAC-PUB-3015 (1982), and is based on the papers Phys. Rev. Lett 45, 1150 (1980) and Phys. Rev. Lett 48, 70 (1982).

The 1S and 2S states of charmonium are observed as very narrow resonances in e^+e^- annihilation. In addition, the 1^{--} 1D state mixes with the S states through a *tensor interaction* of the form

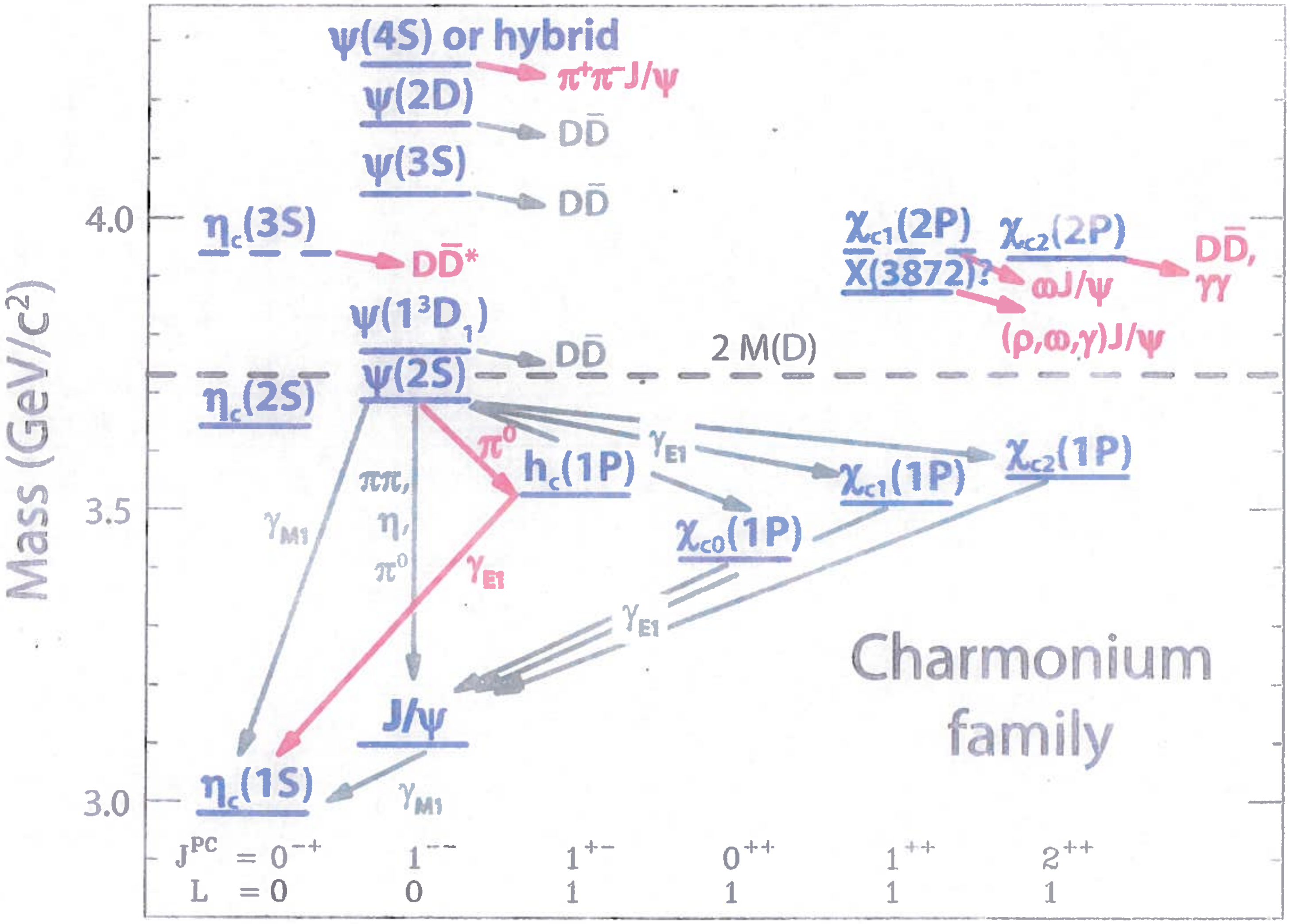
$$\Delta H \sim \left[(\vec{S}\cdot\vec{r})^2 - \frac{1}{3}S^2r^2\right]$$

that is generated by relativistic corrections. This allows the 1D 1^{--} state to appear as an e^+e^- resonance, called the $\psi(3770)$. The 1P 1^{-+} state have been observed as a resonance in proton-antiproton annihilation.

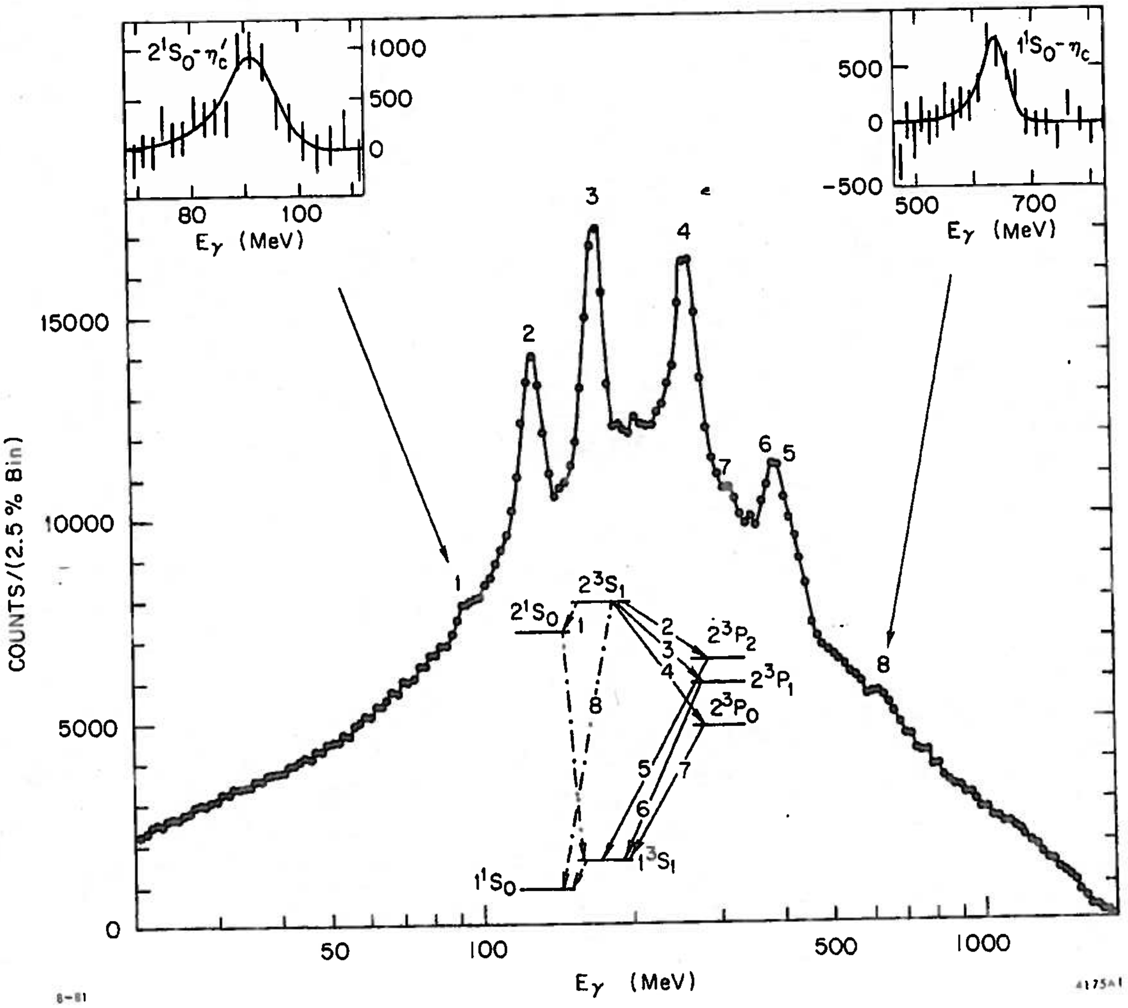
The spectrum of charmonium is further clarified by hadronic transitions such as

$$2S(1^{--}) \to 1S(1^{--}) + \pi^+\pi^-$$

In this process, a pair of pi mesons ($\pi^+\pi^-$) is produced in a 0^{++} state. This can link a pair of state with the same C and P quantum numbers.

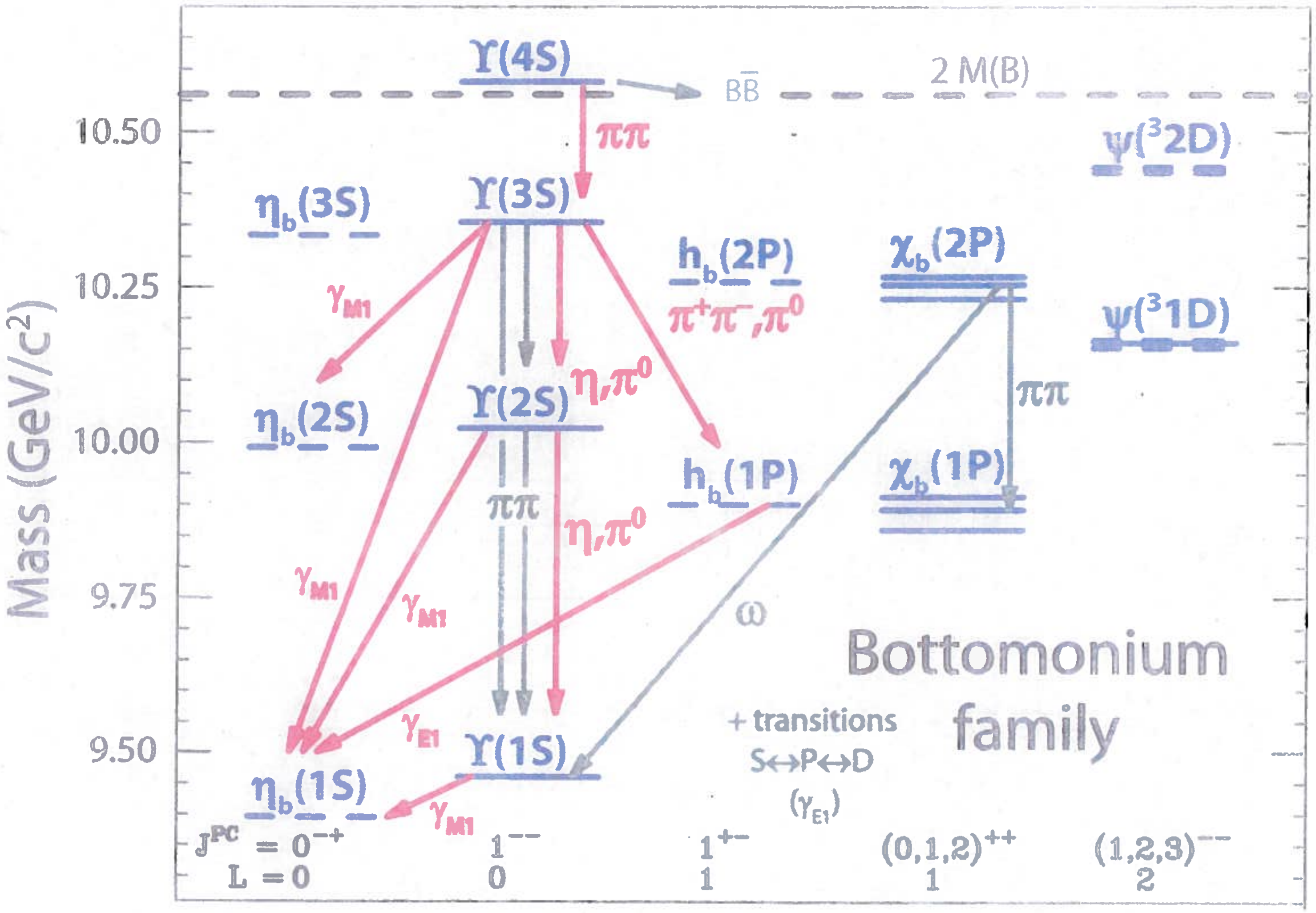


ψ(4S) or hybrid
π⁺π⁻J/ψ
ψ(2D)
DD̄
ψ(3S)
DD̄
η_c(3S)
DD̄*
χ_c1(2P)
X(3872)?
χ_c2(2P)
DD̄,
γγ
ωJ/ψ
(ρ,ω,γ)J/ψ
ψ(1³D₁)
DD̄
2 M(D)
ψ(2S)
η_c(2S)
π⁰
γ_E1
h_c(1P)
χ_c2(1P)
χ_c1(1P)
χ_c0(1P)
γ_M1
ππ,
η,
π⁰
γ_E1
γ_E1
Charmonium
family
J/ψ
η_c(1S)
γ_M1
Mass (GeV/c²)
4.0
3.5
3.0
J^PC = 0⁻⁺
1⁻⁻
1⁺⁻
0⁺⁺
1⁺⁺
2⁺⁺
L = 0
0
1
1
1
1



$2^1S_0 - \eta_c'$
1000
500
0
80
100
E_γ (MeV)
$1^1S_0 - \eta_c$
500
0
-500
500
700
E_γ (MeV)
COUNTS/(2.5 % Bin)
15000
10000
5000
0
50
100
500
1000
E_γ (MeV)
1
2
3
4
5
6
7
8
2^3S_1
2^1S_0
2^3P_2
2^3P_1
2^3P_0
1^1S_0
1^3S_1
8-81
4175A1

The same structure of bound states and transitions is seen for the bound states of the bottom quark and its antiparticle in the energy region around 10 GeV. The known states and their observed transitions – beyond the prominent E1 photon transitions – are shown in the figure, also from the Eichten et al. RMP paper.


Υ(4S)
$B\bar{B}$
2 M(B)
10.50
ππ
ψ(^{3}2D)
η_b(3S)
Υ(3S)
h_b(2P)
χ_b(2P)
10.25
γ_{M1}
$\pi^+\pi^-,\pi^0$
ψ(^{3}1D)
Mass (GeV/c^2)
Υ(2S)
η,π^0
ππ
η_b(2S)
10.00
h_b(1P)
χ_b(1P)
ππ
η,π^0
9.75
γ_{M1}
γ_{M1}
ω
Bottomonium
family
+ transitions
S↔P↔D
(γ_{E1})
γ_{E1}
9.50
Υ(1S)
η_b(1S)
γ_{M1}
J^{PC} = 0^{-+}
L = 0
1^{--}
0
1^{+-}
1
$(0,1,2)^{++}$
1
$(1,2,3)^{--}$
2