May 31

Pion-Nucleon Scattering

Now that we have studied the dynamics of heavy quarks, I would like to discuss some aspects of the dynamics of the lighter quarks that make up the proton and neutron. In Nature, we have 6 types of quark

$$u \quad d \quad s \quad c \quad b \quad t$$

We have seen that the c and b quarks form systems of quarkonium atoms with masses corresponding to

$$m_c c^2 \sim 1.8 \text{ GeV} \qquad m_b c^2 \sim 5. \text{ GeV}$$

Similarly, there are lighter $J^P = 0^-$ states and 1^- states that are interpreted as bound states of the u, d, s quarks. The 1^- states are

$$\begin{array}{cc} \rho^+ \; \rho^0 \; \rho^- & 770 \text{ MeV} \\ \omega^0 & 783 \text{ MeV} \\ K^{+*} K^{0*} \quad \bar{K}^{0*} \; K^{*-} & 892 \text{ MeV} \\ \phi^0 & 1020 \text{ MeV} \end{array}$$

These 9 states are readily interpreted as the 1S $S = 1$ bound states of the lighter quarks and antiquarks, assigning the masses

$$m_u \cong m_d \approx 370 \text{ MeV} \qquad m_s \approx 500 \text{ MeV}$$

There are also 9 0^- states, corresponding to the 1S, $S = 0$ bound states. These have the masses

$$\begin{array}{ll} \pi^+ \; \pi^0 \; \pi^- & 135 \text{ MeV} \\ K^+ \; K^0 \; \bar{K}^0 \; K^- & 495 \text{ MeV} \\ \eta & 547 \text{ MeV} \\ \eta' & 958 \text{ MeV} \end{array}$$

Except for the η', all of these states are unusually light compared to the 1^- states. This property follows from a more subtle aspect of the strong interaction Hamiltonian that I will not explain here. The pattern of electric charges of these states is explained by the charge assignments

$$q(u) = q(d) + e = q(s) + e$$

The quarks also form bound states of a different kind, with 3 quarks binding together to form a fermion. These are called *baryons*. The lowest states of this kind are

$$\begin{array}{ll} p \; n & 938, 939 \text{ MeV} \\ \Delta^{++} \; \Delta^+ \; \Delta^0 \; \Delta^- & 1232 \text{ MeV} \end{array}$$

The proton and neutron, collectively called *nucleons* N, have spin $\frac{1}{2}$, and we take the convention of assigning them parity $P = +1$. The Δ particles have $J = \frac{3}{2}$, $P = +$. The main goal of this lecture is to explain how we know this.

This pattern of bound states is most easily interpreted starting from the Δ^{++}. This is an S-wave bound state of three quarks, all of the same type and all with the same spin orientation. For example,

$$u\uparrow \; u\uparrow \; u\uparrow$$

This requires the odd charge assignments

$$q(u) = +\frac{2}{3}e \qquad q(d) = q(s) = -\frac{1}{3}e$$

but these turn out to be correct and can be independently verified.

Spin $\frac{1}{2}$ particles are fermions, so it is odd that this states is totally symmetric in the spin and orbital quantum numbers. To solve the problem, a number of authors postulated a new quantum number, called *color*, in which the quarks in a Δ are antisymmetric. Eventually, color turned out to be the charge to which the fundamental carriers of the strong interactions, the gluons, couple. We can then postulate that all baryons are completely antisymmetric in color and completely symmetric in all other quantum numbers. The u and d quarks each have 2 spin states, so, in all, the u and d comprise 4 states. The number of totally symmetric states of 3 of these quarks is

$$\frac{4 \cdot 5 \cdot 6}{3!} = 20$$

This corresponds precisely to

$$\begin{array}{llll} 2 \times \text{spin } \frac{1}{2} & = & 4 \text{ states} & (p, n) \\ 4 \times \text{spin } \frac{3}{2} & = & 16 \text{ states} & (\Delta^{++}\ \Delta^{+}\ \Delta^{0}\ \Delta^{-}) \end{array}$$

The number of totally symmetric states of u, d, and s is

$$\frac{6 \cdot 7 \cdot 8}{3!} = 56$$

forming 8 spin $\frac{1}{2}$ and 10 spin $\frac{3}{2}$ states. The heavier states of the multiplets are known. The last spin $\frac{3}{2}$ state, the Ω^{-}, was discovered to some acclaim in 1962.

The u and d quarks have very similar masses, so one might imagine that the strong interactions have a discrete symmetry

$$u \leftrightarrow d$$

weakly broken by the different electromagnetic couplings of these particles. This would induce a discrete symmetry

$$p \leftrightarrow n$$

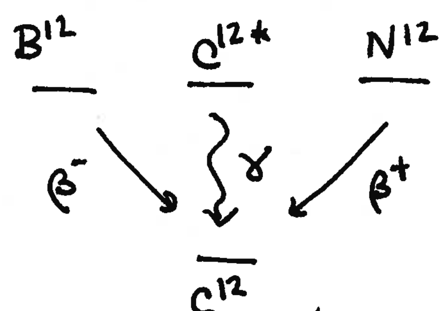
that should be visible in nuclear physics. However, in 1932, Heisenberg made a more ambitious proposal. He suggested that p and n can be thought of as a doublet of a new spin ($SU(2)$) symmetry and that unitary rotations of this doublet should be symmetries of nuclear physics. Today, we would postulate that the strong interactions are invariant under the unitary rotations

$$\begin{pmatrix} u \\ d \end{pmatrix} \to U \begin{pmatrix} u \\ d \end{pmatrix}$$

called *isospin*, and that this induces the transformation

$$\begin{pmatrix} p \\ n \end{pmatrix} \to U \begin{pmatrix} p \\ n \end{pmatrix}$$

which should be a symmetry of nuclear physics. Indeed, it is found that nuclear energy levels form isospin multiplets, for example



Isospin is a more obvious symmetry in elementary particle physics, where small energy separations make less of a difference. If we make the assignment that (u, d) is a doublet of isospin $\frac{1}{2}$, the baryon and meson states that are built as 1S bound states of u and d receive the isospin assignments

$$
\begin{array}{ll}
(p, n) & I = \tfrac{1}{2} \\
(\Delta^{++}\ \Delta^{+}\ \Delta^{0}\ \Delta^{-}) & I = \tfrac{3}{2} \\
(\pi^{+}\ \pi^{0}\ \pi^{-}) & I = 1 \\
\eta & I = 0
\end{array}
$$

If isospin commutes with the strong interaction Hamiltonian, it will give us selection rules and symmetry relations that we can use in analyzing strong interaction scattering processes.

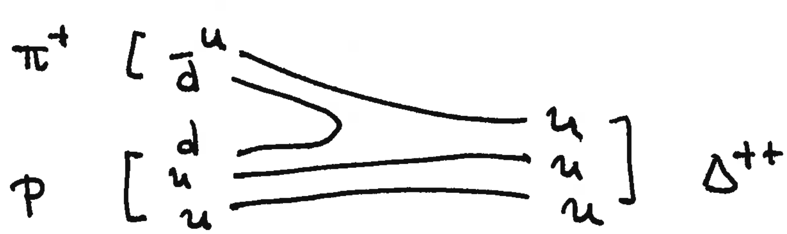
I would now like to work out in detail the consequences of this idea for the scattering of pions from nucleons in the region of the Δ, which should appear as a resonance in this process. We can do this straightforwardly from the formalism that we have built in this course. What we need to do is to write the Fermi Golden Rule formula for scattering through a resonance

$$
\sigma = \frac{1}{V} \int d\Pi \left| \frac{\langle F | \Delta H | R \rangle \langle R | \Delta H | I \rangle}{E_I - E_R + i\Gamma/2} \right|^2
$$

use the symmetries to understand the detailed forms of the matrix elements, and then see what is predicted.

In this analysis, I will consider the nucleon and the Δ as very massive, with a mass difference of 300 MeV, and I will ignore their recoil. The pion mass is 135 MeV, so we will need to treat the pion as a massive but relativistic particle.

For the process we are considering, the cross section formula contains a matrix element between a pion-nucleon state and the Δ state. This matrix element encodes fairly complicated transitions such as



However, this matrix element is strongly constrained by the requirement that it conserve *angular momentum* and *isospin*. Let's first write out the matrix element with all of its dependence on momentum and spin and isospin indices.

$$\langle \Delta(i^3_\Delta, s^3_\Delta) | \, \Delta H \, | \, \pi(\vec{p}, I^3) \, N(i^3, s^3) \rangle$$

Isospin conservation implies that this matrix element is proportional to the Clebsch-Gordon coefficient

$$\langle I^3 \, i^3 \, | \, \tfrac{3}{2} \, i^3_\Delta \rangle$$

Angular momentum conservation requires that the spin of the nucleon and the orbital angular momentum of the pion sum to the spin of the Δ. Then, the orbital angular momentum of the pion is either $L = 1$ or $L = 2$. Since the pion has intrinsic parity $P = -1$ and the Δ has intrinsic parity $P = +1$, we must have another contribution $P = -1$ from the orbital wavefunction. Then, $L = 1$. Angular momentum conservation then requires that the matrix element depend on the spin variables and the momentum of the pion according to

$$Y_{1m}(\hat{p}) \, \langle m \, s^3 \, | \, \tfrac{3}{2} \, s^3_\Delta \rangle$$

At this point, we have determined the dependence of the matrix element on all of its variables. The required form is

$$\langle \Delta(i^3_\Delta, s^3_\Delta) | \Delta H | \pi(\vec{p}, I^3)\ N(i^3, s^3)\rangle$$

$$= g_{\pi N \Delta}\ Y_{lm}(\hat{p})\ \langle m s^3 | \tfrac{3}{2} s^3_\Delta \rangle \langle I^3 i^3 | \tfrac{3}{2} i^3_\Delta \rangle$$

where $g_{\pi N\Delta}$ is an overall constant.

This constant determined the strength of the interaction, which is reflected, in particular in the width of the Δ. It will be useful to compute that width. To do this, we can use the general Fermi Golden Rule formula

$$\Gamma = \int d\Pi\ \left| \langle \pi N | \Delta H | \Delta \rangle \right|^2$$

Since the pion is relativistic, we need to recompute its phase space. The phase space integral is

$$\int d\Pi = \int \frac{d^3p}{(2\pi)^3}\ 2\pi\ \delta(E(p) - (M_\Delta - M_N)c^2)$$

where

$$E(p) = \left[(pc)^2 + (mc^2)^2\right]^{\frac{1}{2}}$$

Going to spherical coordinates,

$$\int d\Pi = \int \frac{d\Omega}{4\pi} \int \frac{dp\ p^2}{\pi}\ \delta(E(p) - (M_\Delta - M_N)c^2)$$

The integral over the delta function is

$$\left[\frac{dE(p)}{dp}\right]^{-1} = \left(\frac{pc^2}{[(pc)^2+(mc^2)^2]^{\frac{1}{2}}}\right)^{-1} = \frac{E}{pc^2}$$

Then, finally,

$$\int d\Pi = \frac{pE}{\pi c^2}\int\frac{d\Omega}{4\pi}$$

This does go properly to the limits that we studied earlier

$$\frac{mp}{\pi}\quad , \quad \frac{p^2}{\pi c}$$

in the limits $p \ll mc$, $p \gg mc$, respectively.

For an initial Δ with isospin i^3_Δ and spin s^3_Δ, we then find

$$I = \frac{pE}{\pi c^2}\sum_{I^3 i^3 s^3}\int\frac{d\Omega}{4\pi}\, g^2_{\pi N\Delta}\left|\sum_m Y^*_{1m}(\hat{p})\,\langle m s^3|\tfrac{3}{2}s^3_\Delta\rangle\langle I^3 i^3|\tfrac{3}{2}i^3_\Delta\rangle\right|^2$$

We can simplify this using

$$\int d\Omega\; Y^*_{1m}(\hat{p})\, Y_{1m'}(\hat{p}) = \delta_{mm'}$$

to give

$$I = \frac{pE\, g^2_{\pi N\Delta}}{4\pi^2 c^2}\cdot\sum_{ms^3}\left|\langle m s^3|\tfrac{3}{2}s^3_\Delta\rangle\right|^2\cdot\sum_{I^3 i^3}\left|\langle I^3 i^3|\tfrac{3}{2}i^3_\Delta\rangle\right|^2$$

The sums in this expression are completeness sums and evaluate to 1. Then

$$\Gamma = \frac{g_{\pi N \Delta}^2 \, p E}{4\pi^2 c^2} \quad \text{with} \quad E = (M_\Delta - M_N) c^2$$

It is a nice check that the width of the Δ is independent of the isospin and spin orientation, as required by rotational invariance.

The Δ^{++} can decay only to $\pi^+ p$, but other Δ states can decay to different pion-nucleon combinations. The branching ratios are controlled by the Clebsch-Gordan coefficients

$$\langle 0 \tfrac{1}{2} | \tfrac{3}{2} \tfrac{1}{2} \rangle = \sqrt{\tfrac{2}{3}} \qquad \langle 1 \, {-\tfrac{1}{2}} | \tfrac{3}{2} \tfrac{1}{2} \rangle = \sqrt{\tfrac{1}{3}}$$

We can read off the pattern

$$\begin{array}{rcl}
\Gamma(\Delta^{++} \to \pi^+ p) & \sim & 1 \\
\Gamma(\Delta^+ \to \pi^+ n),\ \Gamma(\Delta^+ \to \pi^0 p) & \sim & \tfrac{1}{3},\ \tfrac{2}{3} \\
\Gamma(\Delta^0 \to \pi^0 n),\ \Gamma(\Delta^0 \to \pi^- p) & \sim & \tfrac{2}{3},\ \tfrac{1}{3} \\
\Gamma(\Delta^- \to \pi^- n) & \sim & 1
\end{array}$$

These same matrix elements enter the cross sections for producing the Δ resonance, integrated over the width of the resonance,

$$\sigma = \frac{1}{V} \left| \langle \Delta | \Delta H | \pi N \rangle \right|^2 2\pi \, \delta\left(E(p) - (M_\Delta - M_N) c^2\right)$$

The same Clebsch relations give

$$\sigma(\pi^+ p \to \Delta^{++}) : \sigma(\pi^+ n \to \Delta^+) : \sigma(\pi^0 p \to \Delta^+)$$

and

$$= 1 : \tfrac{1}{3} : \tfrac{2}{3}$$

$$\sigma(\pi^+ p \to \Delta^{++}) : \sigma(\pi^- p \to \Delta^0) = 1 : \tfrac{1}{3}$$

This regularity is observed experimentally.

Now we can go back and build the complete formula for the pion-nucleon cross section. Using

$$v = \frac{pc^2}{E}$$

we can rewrite the prefactor as

$$\frac{1}{v}\int d\Pi = \frac{E^2}{\pi c^4}\int \frac{d\Omega}{4\pi}$$

Then we find

$$\sigma = \frac{E^2}{\pi c^4}\int \frac{d\Omega}{4\pi}\,(g^2_{\pi N \Delta})^2 \cdot \left| \sum_{s^3_\Delta i^3_\Delta \bar{m} m} \frac{\langle \bar{m}\, \bar{s}^3 | \tfrac{3}{2} s^3_\Delta \rangle \langle \bar{I}^3 \bar{i}^3 | \tfrac{3}{2} i^3_\Delta \rangle \langle m s^3 | \tfrac{3}{2} s^3_\Delta \rangle \langle I^3 i^3 | \tfrac{3}{2} s^3_\Delta \rangle Y^*_{1\bar{m}}(\hat{p})\, Y_{1m}(\hat{k})}{E_\pi(p) - (M_\Delta - M_N)c^2 + i\Gamma/2} \right|^2$$

There is a lot of physics in this formula. I will analyze it piece by piece.

First, I will work out its implications for the pion angular distribution. Imagine that we set up the scattering experiment so that pions come in along the $\hat{3}$ axis. Assume that the initial nucleon polarization is $s^3 = +\frac{1}{2}$. (We will average over the nucleon spin orientation later.)

In the formula for the cross section $\hat{k}$ is the initial pion direction, along the $\hat{3}$ axis. At this point, $Y_{10}(\hat{k})$ is nonzero, but $Y_{1\pm 1}(\hat{k}) = 0$. Then the initial value of the orbital angular momentum quantum number is $m = 0$. In the final state, we may have $(\overline{m}, \overline{s}^3) = (0, +\frac{1}{2})$ (spin non-flip) or $(\overline{m}, \overline{s}^3) = (1, -\frac{1}{2})$ (spin non-flip). The amplitudes for these two final states should be added *incoherently*, with the squares of their appropriate Clebsches. The processes give

$$\pi^+ p(\uparrow) \to \pi^+ p(\uparrow): \quad |Y_{10}(\hat{p})|^2 |\langle 0 \tfrac{1}{2} | \tfrac{3}{2} \tfrac{1}{2} \rangle|^2 = \frac{3}{4\pi} \cos^2\theta \cdot \frac{2}{3}$$

$$\pi^+ p(\uparrow) \to \pi^+ p(\downarrow) \quad |Y_{11}(\hat{p})|^2 |\langle 1 -\tfrac{1}{2} | \tfrac{3}{2} \tfrac{1}{2} \rangle|^2 = \frac{3}{8\pi} \sin^2\theta \, \frac{1}{3}$$

The overall angular distribution is then

$$\frac{2}{4\pi} \cos^2\theta + \frac{1}{8\pi} \sin^2\theta = \frac{1}{8\pi}(4\cos^2\theta + 1 - \cos^2\theta)$$

$$= \frac{1}{8\pi}(1 + 3\cos^2\theta)$$

To obtain the unpolarized angular distribution, we average this result with that for initial $s^3 = -\frac{1}{2}$. However, you can easily check that that calculation gives the same result. So

$$\frac{d\sigma}{d\cos\theta} \sim (1 + 3\cos^2\theta)$$

This angular distribution is characteristic of the spin $\frac{3}{2}$ character of the Δ resonance. A resonance of spin $\frac{1}{2}$ or opposite parity would give a distinctly different angular distribution. This is the one that is favored by experiment.

The isospin Clebsches determine the magnitudes of the cross sections to various final states. Note that we have a separate squared Clebsch-Gordon coefficient of the type that appeared in the width formulae for the initial state and for the final state. Then, if

$$\frac{d\sigma}{d\cos\theta}(\pi^+ p \to \pi^+ p) \sim 1^2$$

Other amplitudes are smaller by the product of factors from our earlier discussion. For example

$$\frac{d\sigma}{d\cos\theta}(\pi^+ n \to \pi^+ n) = \frac{1}{9}\,\frac{d\sigma}{d\cos\theta}(\pi^+ p \to \pi^+ p)$$

$$\frac{d\sigma}{d\cos\theta}(\pi^+ n \to \pi^0 p) = \frac{2}{9}\,\frac{d\sigma}{d\cos\theta}(\pi^+ p \to \pi^+ p)$$

$$\frac{d\sigma}{d\cos\theta}(\pi^- p \to \pi^- p) = \frac{1}{9}\,\frac{d\sigma}{d\cos\theta}(\pi^+ p \to \pi^+ p)$$

These relations are also in accord with experiment.

Finally, I would like to recast the expression for the cross section in a form that connects to analyses that we did earlier in the course. To do this, I will write σ in the form

$$\sigma = \int d\Omega\, |f|^2$$

pulling the prefactor into the scattering amplitude f. We have

$$f = \frac{E}{2\pi c^2}\, g_{\pi N\Delta}^2 \sum_{s^3_\Delta i^3_\Delta m \bar{m}} \frac{\langle \bar{m}\bar{s}^3 | \tfrac{3}{2} s^3_\Delta\rangle \langle \bar{t}\bar{i}^3 | \tfrac{3}{2} i^3_\Delta\rangle \langle m s^3 | \tfrac{3}{2} s^3_\Delta\rangle \langle t i^3 | \tfrac{3}{2} i^3_\Delta\rangle^*}{E_\pi(p) - (M_\Delta - M_N)c^2 + i\Gamma/2}\, Y_{1\bar{m}}(\hat{p})\, Y_{1m}(\hat{k}$$

We can recognize the ingredients in the Δ resonance width Γ. Then

$$f = \sum_{\substack{s^3_\Delta i^3_\Delta \\ \bar{m} m}} 2\pi \frac{1}{p} \Gamma \frac{\langle I^3 \bar{i}^3 | \frac{3}{2} i^3_\Delta \rangle \langle I^3 i^3 | \frac{3}{2} i^3_\Delta \rangle \langle \bar{m} \bar{s}^3 | \frac{3}{2} s^3_\Delta \rangle \langle m s^3 | \frac{3}{2} s^3_\Delta \rangle}{E_\pi(p) - (M_\Delta - M_N)c^2 + i\Gamma/2} \cdot Y^*_{1\bar{m}}(\hat{p}) Y_{1m}(\hat{k})$$

Now let us remember that the $Y_{1m}(\hat{k})$ is nonzero only for $m = 0$, and, for clarity, let us pick out the term with $\overline{m} = 0$. This gives

$$f = \sum_{s^3_\Delta i^3_\Delta} \frac{2\pi}{p} \Gamma \frac{3}{4\pi} \cos\theta \frac{\langle I^3 \bar{i}^3 | \frac{3}{2} i^3_\Delta \rangle \langle I^3 i^3 | \frac{3}{2} i^3_\Delta \rangle \langle 0 \bar{s}^3 | \frac{3}{2} s^3_\Delta \rangle \langle 0 s^3 | \frac{3}{2} s^3_\Delta \rangle}{E_\pi(p) - (M_\Delta - M_N)c^2 + i\Gamma/2}$$

or

$$f = \frac{3}{p} \left[\frac{\Gamma/2}{E_\pi(p) - (M_\Delta - M_N)c^2 + i\Gamma/2} \right] \cos\theta \sum_{s^3_\Delta i^3_\Delta} \langle 1 \rangle \langle 1 \rangle \langle 0 \bar{s}^3 | \tfrac{3}{2} s^3_\Delta \rangle \langle 0 s^3 | \tfrac{3}{2} s^3_\Delta \rangle$$

We recognize the factor in brackets as $e^{i\delta} \sin\delta$. Adding back the term with $\overline{m} = 1$, we see that the complete scattering amplitude takes the form

$$f = \frac{3}{p} e^{i\delta} \sin\delta_\ell \sum_{s^3_\Delta} \left[\cos\theta \langle 0 \bar{s}^3 | \tfrac{3}{2} s^3_\Delta \rangle + \frac{1}{\sqrt{2}} \sin\theta e^{i\phi} \langle 1 \bar{s}^3 | \tfrac{3}{2} s^3_\Delta \rangle \right] \cdot \langle 0 s^3 | \tfrac{3}{2} s^3_\Delta \rangle \cdot \sum_{i^3_\Delta} \langle I^3 \bar{i}^3 | \tfrac{3}{2} i^3_\Delta \rangle \langle I^3 i^3 | \tfrac{3}{2} i^3_\Delta \rangle$$

The first term has just the structure

$$f \sim \frac{2\ell + 1}{p} e^{i\delta} \sin\delta_\ell P_\ell(\cos\theta)$$

with $\ell = 1$, times Clebsch-Gordon coefficients, and the second term generalizes this appropriately for spin-flip scattering.

This is very suggestive. We find an expression of the form that we saw earlier in partial wave analysis, parametrized by a phase shift, together with Clebsch-Gordan coefficients that project the initial and final states onto the resonant channel. In a full analysis of pion-nucleon scattering, the amplitude would be a sum of terms of this type, one for each possible value of the quantum numbers J, L, and I of the resonance.

Notice the factor

$$\langle 0\, s^3 | \tfrac{3}{2}\, s^3_\Delta \rangle$$

projecting the initial spin state onto the $J = \frac{3}{2}$ channel. This tells us that the resonance is only

$$\left| \langle 0 \pm\tfrac{1}{2} | \tfrac{3}{2} \pm\tfrac{1}{2} \rangle \right|^2 = \frac{2}{3}$$

times as high as it would be if, in a perfect world, we could create a $J = \frac{3}{2}$ initial state. The final state spin Clebsches give the angular distribution, as we computed it above, and the isospin Clebsches give the factors for the various initial and final isospin states.

The figure, taken from the *Review of Particle Properties* of the LBL Particle Data Group, shows the latest compilation of data on the total and elastic cross sections for pi^+p and π^-p scattering. The Δ resonance is the large structure on the left side of the plot. Note that the π^+p cross section at the Δ peak is 3 times higher than the π^-p total cross section, which is in turn 3 times higher than the π^-p elastic cross section. This is in accord with our calcuations above.

At higher energies, we see additional resonances. The first is the $N^*(1518)$, which turns out to have $I = \frac{1}{2}$, $J^P = \frac{3}{2}^-$. There is also a smaller resonance, not so visible on this plot, the $N^*(1440)$, with $I = \frac{1}{2}$, $J^P = \frac{1}{2}^+$. These are P and S wave orbitally excited states of the proton. The quantum numbers of these and higher resonances can be obtained by extension of the analysis in this lecture. Eventually, though, the resonances merge together to form an approximately constant cross section at very high energies. This cross section is better understood using the diffractive models discussed earlier in the course.

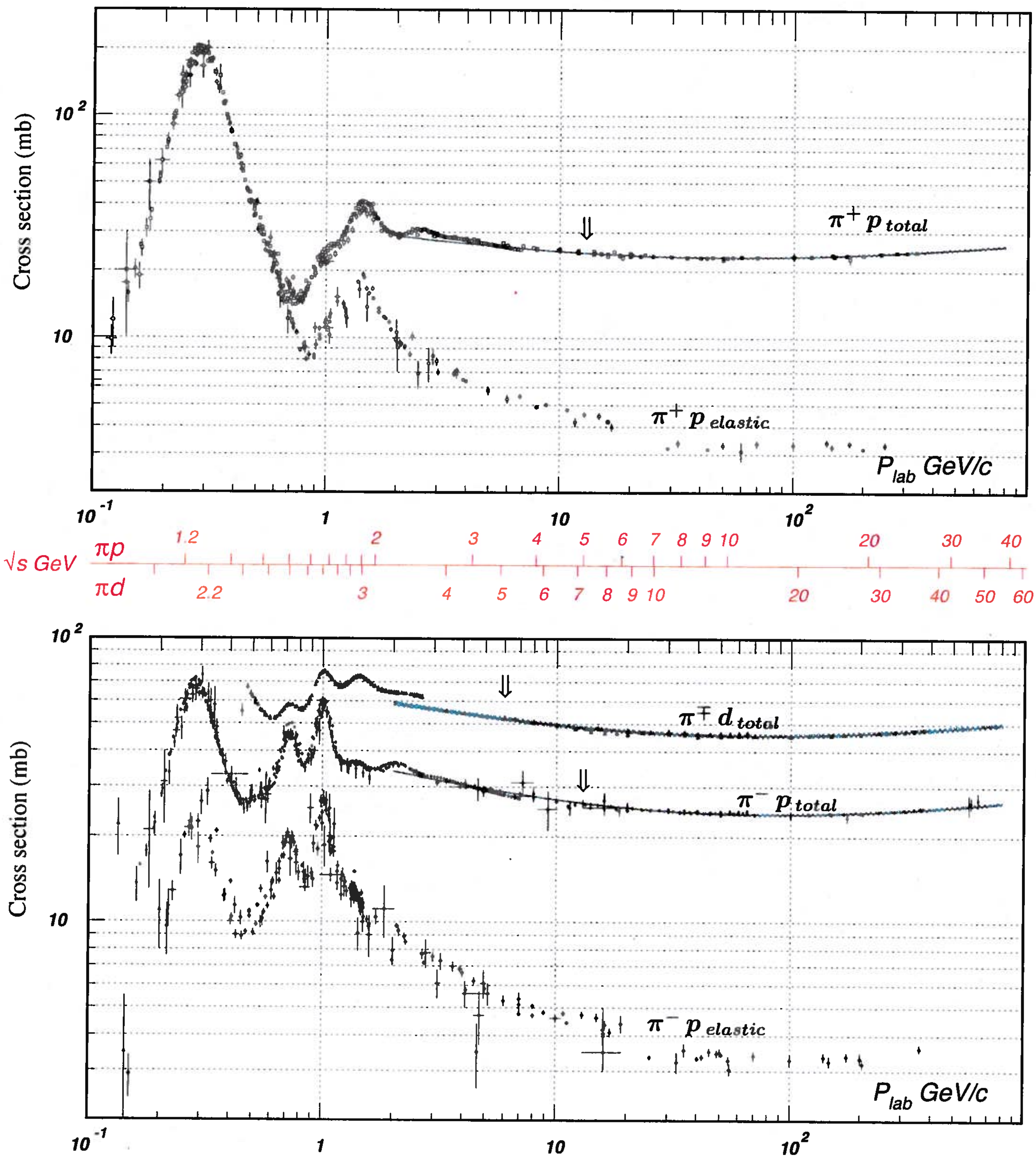


Figure 41.13: Total and elastic cross sections for $\pi^{\pm} p$ and $\pi^{\pm} d$ (total only) collisions as a function of laboratory beam momentum and total center-of-mass energy. Corresponding computer-readable data files may be found at `http://pdg.lbl.gov/current/xsect/`. (Courtesy of the COMPAS Group, IHEP, Protvino, August 2005)