# Physics 134 Final Exam Solutions

1.) The expression for the scattering amplitude is

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi}{k^2}\left| 3\cos\theta\,\left(e^{i\delta_1}\sin\delta_1\right) + 5\,\frac{3\cos^2\theta-1}{2}\left(e^{i\delta_2}\sin\delta_2\right)\right|^2$$

where $\delta_1$ goes through $\pi/2$ in a region $\pm 4$ MeV about 20 MeV
$\delta_2$ goes through $\pi/2$ in a region $\pm 1$ MeV about 20 MeV

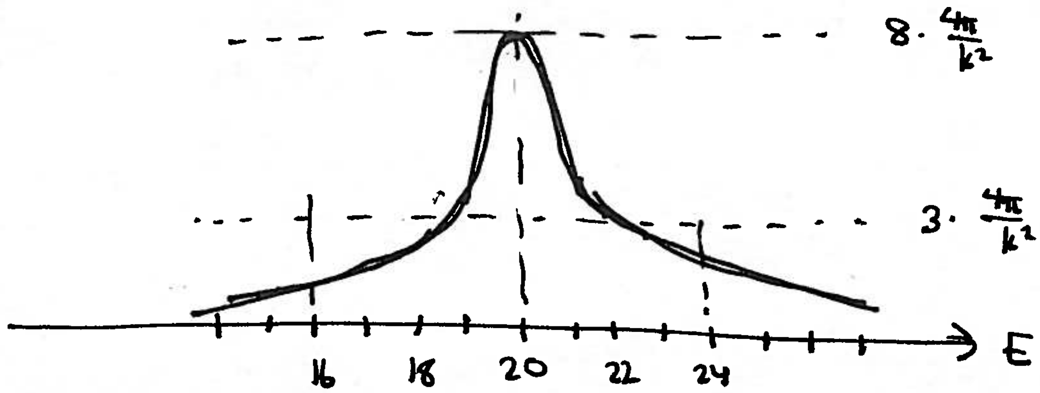
a) The total cross section is

$$\sigma = \frac{4\pi}{k^2}\left(3\sin^2\delta_1 + 5\sin^2\delta_2\right)$$

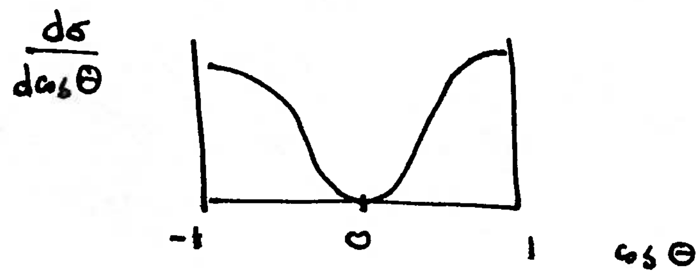
at 20 MeV $\quad \delta_1 = \delta_2 = \pi/2 \quad$ so

$$\sigma = \frac{32\pi}{k^2}$$

b) Note the widths of the resonances:




c) at 15 MeV only the $l=1$ resonance is present



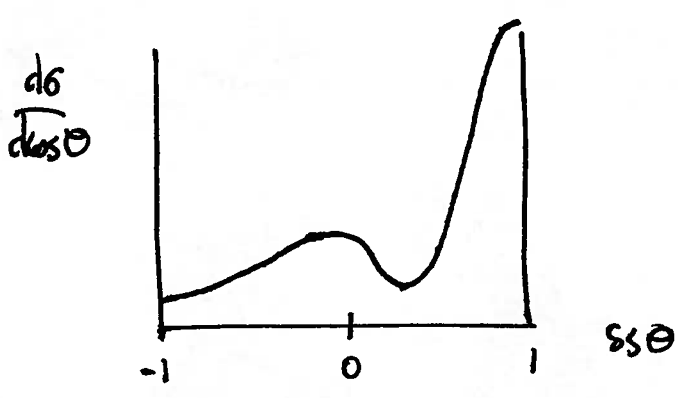

at 19 MeV we can approximate

$$e^{i\delta_1}\sin\delta_1 \approx i$$

$$e^{i\delta_2}\sin\delta_2 \approx \frac{1+i}{2}$$

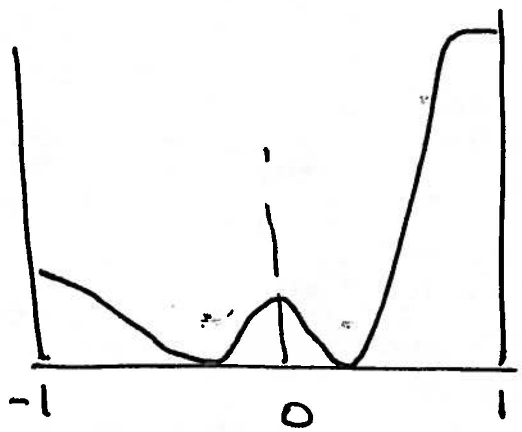
the imaginary, interfering, part of the amplitude is

$$3i\cos\theta + \frac{5i}{4}\cdot(3\cos^2\theta - 1) = \begin{cases} 5\frac{1}{2}\, i & \theta = 0 \\ -\frac{5}{4}\, i & \theta = \pi/2 \\ -\frac{1}{2}\, i & \theta = \pi \end{cases}$$



at 20 MeV the amplitude is

$$3\cos\theta + \frac{5}{2}(3\cos^2\theta - 1) = \begin{cases} 8 & \text{at } \theta = 0 \\ -5/4 & \pi/2 \\ 2 & \pi \end{cases}$$




there are two zeros.

at 21 MeV, the situation is similar to that at 19 MeV

we now have

$$e^{i\delta_2}\sin\delta_2 \approx \frac{-1+i}{2}$$

so the real part of the amplitude flips sign, but this does not affect $\frac{d\sigma}{d\cos\theta}$

21 MeV $\longrightarrow$ same as 19 MeV

and 25 MeV $\longrightarrow$ same as 15 MeV

2) a. $$\mathcal{H} = -g\frac{q}{2m}\vec{S}\cdot\vec{B}$$

This is a 2-state system $\qquad S^3 = \frac{\sigma^3}{2}$

a) the energies are

$$\downarrow \quad E = +\frac{gqB}{4m}$$

$$\uparrow \quad E = -\frac{gqB}{4m}$$

b) the transition amplitude is

$$-i\int_0^t dt' \, \langle\downarrow| e^{iH_0t'}\left(-\frac{gq}{2m}S^1B^1(t')\right)e^{-iH_0t'}|\uparrow\rangle$$

$$S^1 = \frac{\sigma^1}{2} \qquad B^1(t') = bt'$$

$$= -i\int_0^t dt' \; e^{i\frac{gq}{4m}Bt'}\left(-\frac{gq}{4m}bt'\right)e^{i\frac{gqB}{4m}t'}$$

$$= +\int_0^t dt' \left(i\frac{gqb}{4m}t'\right) e^{i\frac{gqB}{2m}t'}$$

$$= \frac{b}{2}\frac{\partial}{\partial B}\int_0^t dt' \; e^{i\frac{gqB}{2m}t'}$$

$$= \frac{b}{2}\frac{\partial}{\partial B}\left(\frac{e^{i\frac{gqB}{2m}t}-1}{i\frac{gqB}{2m}}\right)$$

$$= \frac{b}{2}\left(i\,\frac{gq}{2m}\right)\frac{1}{\left(\frac{igqB}{2m}\right)^2}\left[-\left(e^{i\frac{gqB}{2m}t}-1\right)+t\,e^{i\frac{gqB}{2m}t}\cdot\left(\frac{igqB}{2m}\right)\right]$$

$$= \frac{b}{2B}\left[t\,e^{i\frac{gqB}{2m}t} - \frac{\left(e^{i\frac{gqB}{2m}t}-1\right)}{\left(\frac{igqB}{2m}\right)}\right]$$

c.) For $b \ll 1$, the amplitude grows like $t$ ~~not~~ $t^2$. Because the $\uparrow$ and $\downarrow$ states get out of phase with one another, the transition cannot be driven at full strength for times larger than $\left(\frac{gqB}{2m}\right)^{-1}$.

d) For small $b$, the second term above is always small. The first term becomes of order 1 when

$$\frac{bt}{2B} \sim 1$$

Then 1st order time-dep. perturbation theory breaks down. So, this happens at

$$t \sim \frac{2B}{b}$$

3.) a.) The $L=2$ $J=3/2$ states with $J^3>0$ are

$$|J=3/2 \;\; J^3=3/2\rangle = \sqrt{\tfrac{4}{5}}\,|m=2 \;\; s^3=-\tfrac{1}{2}\rangle + \left(-\sqrt{\tfrac{1}{5}}\right)|m=1 \;\; s^3=\tfrac{1}{2}\rangle$$

$$|J=3/2 \;\; J^3=1/2\rangle = \sqrt{\tfrac{3}{5}}\,|m=1 \;\; s^3=-\tfrac{1}{2}\rangle + \left(-\sqrt{\tfrac{2}{5}}\right)|m=0 \;\; s^3=\tfrac{1}{2}\rangle$$

taking the Clebsch's from the table.

b.) An E1 transition goes via the operator $\vec{\epsilon}\cdot\vec{r}$. This has spin 1, for $|\Delta L|\le 1$. The operator also has odd parity so $L=2 \to L=2$ is forbidden. Then we can have

$$L=3 \text{ or } L=1$$

For $L=3$ we have $J=5/2$ or $J=7/2$

$L=1$ we have $J=1/2$ or $J=3/2$

but, also $|\Delta J|\le 1$ so $J=7/2$ cannot be reached.

so
$$(L=2 \;\; J=3/2) \xrightarrow{E1} (L=3 \;\; J=5/2)$$
$$(L=1 \;\; J=1/2),\ (L=1, J=3/2)$$

c.) If the photon has $\vec{\epsilon}=\vec{\epsilon}_+$, absorbtion of this photon raises $m$. Then the final state must be a state that includes

$$|m=3\ S^3=-\tfrac{1}{2}\rangle \text{ or } |m=2\ S^3=+\tfrac{1}{2}\rangle$$

This is not possible for any $L=1$ state, so only the

$$\underline{L=3\quad J=5/2} \text{ state is seen}$$

d.) In our discussion of E1 transitions we saw that

emission of a photon with polarization $\vec{\epsilon}=\hat{3}$ leads to $\dfrac{d\sigma}{d\cos\theta} \sim \dfrac{3}{4}\sin^2\theta$

emission of a photon with polarization $\vec{\epsilon}=\vec{\epsilon}_\pm$ leads to $\dfrac{d\sigma}{d\cos\theta} \sim \dfrac{3}{8}(1+\cos^2\theta)$

so we just need to work out the various cases for the processes

$$\gamma + (L=2\ J=3/2) \underset{\epsilon}{\rightarrow} (L=3\ J=5/2) \rightarrow \gamma + (L=2\ J=3/2)$$

The amplitudes are proportional to Clebsch-Gordan coefficients for

$$(J=3/2\ ,\ J^3) + (J_\gamma=1\ m_\gamma) \rightarrow (\hat{J}=5/2\quad \hat{J}^3)$$

for the initial step, and the reverse (i.e. the same) for the final step.

Photons come in along the $\hat{3}$ axis, so $\vec{\epsilon} = \vec{\epsilon}_\pm$ only (i.e. $m_\gamma = \pm 1$ only). These two cases, summed over all atomic orientations $J^3$, give the same angular distributions. Each set of transitions then contributes

$$\left|\langle J^3, m_\gamma = +1 \,|\, \tfrac{5}{2}\, \hat{J}^3 \rangle\right|^2 \left|\langle J^{3\prime}\, m_\gamma' \,|\, \tfrac{5}{2}\, \hat{J}^3 \rangle\right|^2 \cdot (\text{angular dist.})$$

given $J^3, m_\gamma$, $\hat{J}^3$ is fixed: $\hat{J}^3 = J^3 + m_\gamma$. There may be several values of $m_\gamma'$ possible, with $J^{3\prime} = \hat{J}^3 - m_\gamma'$. This add incoherently.

Here is the table: Using $\underline{J = \tfrac{3}{2} \times J = 1}$ Clebsches
$\hookrightarrow J = \tfrac{5}{2}$

$J^3 = \tfrac{3}{2} \longrightarrow J^{3\prime} = \tfrac{3}{2}\ \ m_\gamma' = 1$ :
$m_\gamma = +1$

$$1 \cdot 1 \cdot \tfrac{3}{8}(1 + \cos^2\theta)$$

$J^3 = \tfrac{1}{2} \longrightarrow J^{3\prime} = \tfrac{1}{2}\ \ m_\gamma' = 1$
$m_\gamma = +1$

$$\tfrac{3}{5} \cdot \tfrac{3}{5} \cdot \tfrac{3}{8}(1 + \cos^2\theta)$$

$\longrightarrow J^{3\prime} = -\tfrac{3}{2}\ \ m = 0$

$$\tfrac{3}{5} \cdot \tfrac{2}{5} \qquad \tfrac{3}{4}\sin^2\theta$$

$J^3 = -\frac{1}{2} \longrightarrow J^{3\prime} = -\frac{1}{2} \quad m_\gamma' = 1$

$m_\gamma = +1$

$\frac{3}{10} \cdot \frac{3}{10} \cdot \frac{3}{8}(1+\cos^2\theta)$

$\longrightarrow J^{3\prime} = \frac{1}{2} \quad m_\gamma' = 0$

$\frac{3}{10} \cdot \frac{3}{5} \cdot \left(\frac{3}{4}\sin^2\theta\right)$

$\longrightarrow J^{3\prime} = \frac{3}{2} \quad m_\gamma' = -1$

$\frac{3}{10} \cdot \frac{1}{10} \cdot \frac{3}{8}(1+\cos^2\theta)$

$J^3 = -\frac{3}{2} \longrightarrow J'^3 = -\frac{3}{2} \quad m_\gamma' = 1$

$m_\gamma = +1$

$\frac{1}{10} \cdot \frac{1}{10} \quad \frac{3}{8}(1+\cos^2\theta)$

$J'^3 = -\frac{1}{2} \quad m_\gamma' = 0$

$\frac{1}{10} \cdot \frac{3}{5} \quad \frac{3}{4}\sin^2\theta$

$J^3 = \frac{1}{2} \quad m_\gamma' = -1$

$\frac{1}{10} \cdot \frac{3}{10} \quad \frac{3}{8}(1+\cos^2\theta)$

Add the pieces:

$\frac{1}{25} \cdot \frac{3}{8}(1+\cos^2\theta)\left[25 + 9 + \frac{9}{4} + \frac{3}{4} + \frac{1}{4} + \frac{3}{4}\right]$

$+ \frac{1}{25} \frac{3}{4}\sin^2\theta \left[6 + \frac{9}{2} + \frac{3}{2}\right]$

$$= \frac{1}{25}\left(\frac{3}{8}(1+\cos^2\theta)\,[38] + \left(\frac{3}{4}\sin^2\theta\right)[12]\right)$$

$$= \frac{3}{8}\cdot\frac{2}{25}\cdot\left[19\,(1+\cos^2\theta) + 12\,(1-\cos^2\theta)\right]$$

$$= \quad + \quad [\,31 - 7\cos^2\theta\,]$$

$$\left.\frac{d\sigma}{d\cos\theta}\right|_{\text{unpol.}} \sim (31 - 7\cos^2\theta)$$

4.) a.) For $\langle | \Delta H | \rangle = A \cdot \underline{1}$

$$P(T) = P(P) \cdot P(\nu)$$

then $P(\nu) = -1$

For $\langle | \Delta H | \rangle = B\,[\hat{p} \cdot \vec{\sigma}]$

parity must reverse the sign of the matrix element, since it will be evaluated at $(-\hat{p})$ Then

$$P(T) = -P(P)\,P(\nu) \quad \text{and} \quad P(\nu) = +1$$

b.) Let $\Delta E = (m_T c^2 - m_p c^2)$ $\qquad p_\nu = \frac{\Delta E}{c}$

$$I = \int d\Pi \sum_{\text{spins}} \left| \langle P\nu | \Delta H | T \rangle \right|^2 \cdot \frac{1}{2}$$

spin avg. for T

$$\int d\Pi = \frac{p_\nu^2}{\pi c} \int \frac{d\Omega}{4\pi}$$

$$I = \frac{1}{2} \frac{p_\nu^2}{\pi c} \int \frac{d\Omega}{4\pi} \sum_{s^3 s_T^3 = \pm 1/2} \left( A\,\delta_{s^3 s_T^3} + B(\hat{p} \cdot \vec{\sigma})_{s^3 s_T^3} \right) *$$

$$\cdot \left( A^* \delta_{s^3 s_T^3} + B^* (\hat{p} \cdot \vec{\sigma})_{s_T^3 s^3} \right)$$

then the spin sum is

$$2 \cdot |A|^2 + (A^*B + B^*A)\, \mathrm{tr}\,[\hat{p}\,\vec{\sigma}] + |B|^2\, \mathrm{tr}\,(\hat{p}\,\vec{\sigma})^2$$

Now $\hat{p}\,\vec{\sigma}$ is a Pauli sigma matrix like $\sigma^1$ Notice that

$$\mathrm{tr}\,\sigma^1 = 0 \qquad \mathrm{tr}\,(\sigma^1)^2 = \mathrm{tr}\,1 = 2$$

This applies for general $\hat{p}\cdot\vec{\sigma}$

$$= 2\,|A|^2 + 2\,|B|^2$$

thus

$$I = \frac{p^2 v}{\pi c}\left(|A|^2 + |B|^2\right)$$

c) If the T is polarized, we need to fix $s_T^3 = +\frac{1}{2}$, but we still sum over $s^3$

$$I = \frac{p^2 v}{\pi c}\int \frac{d\Omega}{4\pi}\, |A|^2\, \left[(1-\hat{p}\,\vec{\sigma})(1-\hat{p}\,\vec{\sigma})\right]_{s_T^3 s_T^3}$$

evaluated for $s_T^3 = +\frac{1}{2}$

$$= \frac{p^2 v}{\pi c}\int \frac{d\Omega}{4\pi}\, |A|^2 \left[1 - 2(\hat{p}\,\vec{\sigma})_{\frac{1}{2}\frac{1}{2}} + (\hat{p}\,\vec{\sigma})^2_{\frac{1}{2}\frac{1}{2}}\right]$$

again $(\hat{p}\vec{\sigma})^2 = 1$ so $([\hat{p}\vec{\sigma}]^2)_{\uparrow\uparrow} = 1$

$$\hat{p}\vec{\sigma} = \begin{pmatrix} \cos\theta & \sin\theta e^{i\phi} \\ \sin\theta e^{-i\phi} & -\cos\theta \end{pmatrix} \quad \text{for } \hat{p} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

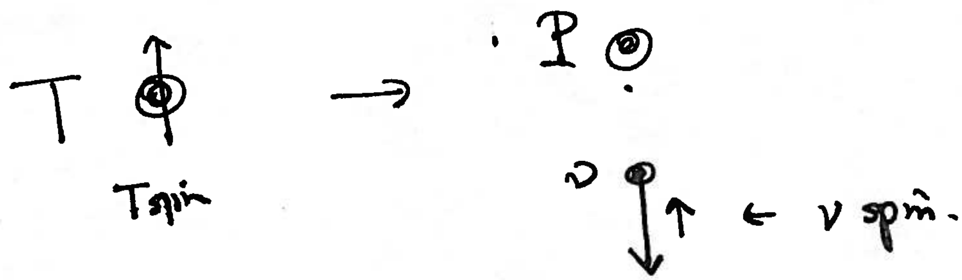
so $(\hat{p}\vec{\sigma})_{\uparrow\uparrow} = \cos\theta$

in all

$$I = \frac{p_\nu^2}{\pi c}\int \frac{d\Omega}{4\pi} |A|^2 \cdot 2(1-\cos\theta)$$

$$\frac{dI}{d\cos\theta} = \frac{p_\nu^2 |A|^2}{\pi c} \cdot (1-\cos\theta)$$

This actually makes sense. The matrix $(1-\hat{p}\vec{\sigma})$ projects the $\nu$ spin to be opposite to $\hat{p}$. Then the preferred configuration is




conserving angular momentum.

d.) Momentum $\vec{p}$ is odd under parity, $\vec{S}$ is even

In this decay

$$\langle \vec{S}_T \cdot \vec{P}_\nu \rangle < 0$$

But

$$\mathcal{P} ( \vec{S}_T \cdot \vec{P}_\nu ) \mathcal{P}^{-1} = - \vec{S}_T \cdot \vec{P}_\nu$$

so in a parity conserving theory we must have

$$\langle \vec{S}_T \cdot \vec{P}_\nu \rangle = 0$$

e.) $$I = \frac{p_\nu^2}{\pi c} \int \frac{d\Omega}{4\pi} |C|^2$$

$$\cdot \frac{1}{2} \sum_{s^3 s_T^3 \epsilon} \left| \left[ (1 - \hat{p} \cdot \vec{\sigma}) \vec{\epsilon} \cdot \vec{\sigma} \right]_{s^3 s_T^3} \right|^2$$

the second line can be rewritten

$$\frac{1}{2} \sum_{\epsilon} \mathrm{tr} \left[ (1 - \hat{p} \cdot \vec{\sigma}) \vec{\epsilon} \cdot \vec{\sigma} \; \vec{\epsilon} \cdot \vec{\sigma} \; (1 - \hat{p} \cdot \vec{\sigma}) \right]$$

Now, the sum over $\vec{\epsilon}$ is

$$\sum_{\epsilon} \vec{\epsilon} \cdot \vec{\sigma} \; \vec{\epsilon} \cdot \vec{\sigma} = (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 = 3 \cdot \underset{\sim}{1}$$

so the expression becomes

$$\frac{1}{2} \cdot 3 \ \mathrm{tr}\, (1-\hat{p}\cdot\vec{\sigma})^2$$

$$= \frac{3}{2} \mathrm{tr} \left[1 - 2\hat{p}\vec{\sigma} + (\hat{p}\vec{\sigma})^2\right]$$

$$= \frac{3}{2} \cdot (2 - 0 + 2) = 6$$

then

$$I = 6 \frac{p_\nu^2}{\pi c} |C|^2$$

f.) For a T polarized with $S_T^3 = +\frac{1}{2}$ we need to rework the spin sum. Instead of the second line above, we have:

$$\sum_{S^3 \epsilon} \left| \left[(1-\hat{p}\cdot\vec{\sigma})\, \vec{\epsilon}\cdot\vec{\sigma}\right]_{S^3\, S_T^3 = \frac{1}{2}} \right|^2$$

$$= \sum_{\epsilon} \left[\vec{\epsilon}\cdot\vec{\sigma}\, (1-\hat{p}\cdot\vec{\sigma})(1-\hat{p}\cdot\vec{\sigma})\, \vec{\epsilon}\cdot\vec{\sigma}\right]_{\frac{1}{2}\frac{1}{2}}$$

This is not so easy to work out by brute force, but some trickiness can be applied. First

$$(1-\hat{p}\vec{\sigma})(1-\hat{p}\vec{\sigma}) = 1 - 2\hat{p}\vec{\sigma} + (\hat{p}\vec{\sigma})^2 = 2(1-\hat{p}\vec{\sigma})$$

Now we have

$$2 \cdot \sum_{\epsilon} \left[ \vec{\epsilon}\cdot\vec{\sigma} \; (1 - \hat{p}\,\vec{\sigma}) \; \vec{\epsilon}\cdot\vec{\sigma} \right]_{\frac{1}{2}\frac{1}{2}}$$

the first term is simple:

$$\sum_{\epsilon} \left( \vec{\epsilon}\cdot\vec{\sigma} \cdot 1 \cdot \vec{\epsilon}\cdot\vec{\sigma} \right)_{\frac{1}{2}\frac{1}{2}} = 3 \, (\underset{\sim}{1})_{\frac{1}{2}\frac{1}{2}} = 3$$

For the second term choose a basis for the $\vec{\epsilon}$ so that one direction is parallel to $\hat{p}$. For this matrix

$$(\hat{p}\cdot\vec{\sigma})(\hat{p}\,\vec{\sigma})(\hat{p}\cdot\vec{\sigma}) = (\hat{p}\cdot\vec{\sigma})$$

For those $\vec{\epsilon}$ perpendicular to $\hat{p}$ $\vec{\epsilon}_{\perp}\cdot\vec{\sigma}$ anticommutes with $\hat{p}\cdot\vec{\sigma}$

$$\vec{\epsilon}_{\perp}\cdot\vec{\sigma} \; (\hat{p}\cdot\vec{\sigma}) \, (\vec{\epsilon}_{\perp}\cdot\vec{\sigma}) = -\hat{p}\cdot\vec{\sigma}$$

so the above is

$$2 \cdot \left( 3 - (\hat{p}\,\vec{\sigma})_{\frac{1}{2}\frac{1}{2}} \right) = 2 \, ( 3 - \cos\Theta )$$

$$\frac{d\Gamma}{d\cos\Theta} = \frac{p_{\nu}^{2}}{\pi c} \, |C|^{2} \, (3 - \cos\Theta)$$