May 8

Fermi's Golden Rule

We have now completed the discussion of the exact solution to the Wigner-Weisskopf problem. We have looked at this problem from two different points of view, both as a problem in electron scattering and as a problem in the time-dependence of an unstable quantum state, and we have gained understanding from the interrelation of these solutions. However, we have one more task in front of us. Few quantum mechanics problems are exactly solvable, and so it would be good to develop approximation methods that reproduce main features of the exact solution to the Wigner-Weisskopf problem. In particular, many of the nontrivial features of this problem are present already at weak coupling. Can we derive these features by doing perturbation theory in the coupling constant?

To treat this problem, I will first develop a general method for carrying out perturbation theory for the time-dependence of a quantum system. We will see in a moment that it is important to approach questions about time-dependent quantum states in a systematic way.

Consider, then, the problem of integrating the time-dependent Schrödinger equation

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle$$

from a definite initial condition, in a case where the Hamiltonian H splits naturally into an H_0 that is easy to diagonalize and a perturbation,

$$\mathcal{H} = \mathcal{H}_0 + \lambda V$$

In the following, I will take λ to be a small dimensionless parameter and try to develop the solution to the Schrödinger equation in powers of λ.

For the Wigner-Weisskopf problem, H_0 is given by

$$\mathcal{H}_0 |\vec{r}\rangle = -\frac{\nabla^2}{2m} |\vec{r}\rangle \qquad \mathcal{H}_0 |0\rangle = E_0 |0\rangle$$

and V is the perturbation that mixes the state $|0\rangle$ with the free-particle continuum

$$V|0\rangle = \int d^3r \, \chi(r) |\vec{r}\rangle \qquad V|\vec{r}\rangle = \chi^*(\vec{r}) |0\rangle$$

The simplest approach would be to start with the initial condition

$$|\Psi(t=0)\rangle = |0\rangle$$

and integrate the Schrodinger equation systematically. In the notation of the previous lecture, the initial condition is

$$\psi(\vec{r}, t) = 0 \qquad \alpha(t) = 1 \qquad \text{at } t = 0$$

Integrating the Schrödinger equation, we obtain

$$i \frac{\partial}{\partial t} \psi(\vec{r}, t) \sim \lambda \chi(r)$$

or

$$\psi(\vec{r}, t) \sim -i \lambda \chi(\vec{r}) \cdot t$$

This is odd; it leads to a result

$$\int d^3r\, |\psi(\vec{r},t)|^2 \sim \lambda^2 t^2$$

But, from the exact solution given in the previous lecture, we know that

$$\text{Prob}(|0\rangle) = e^{-\Gamma t} \cong 1 - \Gamma t$$

and so

$$\text{Prob}(\text{continuum}) = \Gamma t \sim \lambda^2 t$$

which would seem to contradict the obvious result from integration. To resolve this problem, we need a more systematic and much more clever formalism.

To build this formalism, I will introduce a new *picture* to described the time-dependence of quantum mechanical operators and states. Up to this point in the course, I have worked in the *Schrödinger picture* in which operators are time-independent and states obey the Schrödinger equation

$$i \frac{\partial}{\partial t} |\Psi(t)\rangle = \underset{\sim}{H} |\psi(t)\rangle$$

States then evolve according to

$$|\psi(t)\rangle = e^{-i \underline{H} t} |\psi(0)\rangle$$

Alternatively, it is possible to put the time-dependence on the operators, leaving the states fixed in time. To do this, write

$$|\Psi\rangle = |\Psi(t=0)\rangle \qquad \underset{\sim}{O}(t) = e^{iHt}\, \underset{\sim}{O}\, e^{-iHt}$$

The time-dependence of matrix elements of the operators in the states is unchanged, but now it is the operators rather than the states that vary in time. This approach is called the *Heisenberg picture.* The equation of motion satisfied by operators is

$$i \frac{\partial}{\partial t} \underset{\sim}{O}(t) = [\underset{\sim}{O}(t), \underset{\sim}{H}]$$

For example, if

$$\underset{\sim}{O} = \underset{\sim}{x}, \underset{\sim}{p} \qquad \underset{\sim}{H} = \frac{p^2}{2m} + V(\underset{\sim}{x})$$

then

$$i \frac{\partial}{\partial t} \underset{\sim}{x} = [\hat{x}, \underset{\sim}{H}] = i \frac{p}{m}$$

$$i \frac{\partial}{\partial t} \underset{\sim}{p} = [\hat{p}, \underset{\sim}{H}] = -i \frac{\partial V}{\partial x}$$

The time-dependent operators then satisfy the differential equations

$$\dot{x}(t) = \frac{p}{m} \qquad \dot{p}(t) = -\frac{\partial V}{\partial x}$$

More generally, in the Heisenberg picture, operators satisfy equations of motion similar to the Hamiltonian equations of classical mechanics. This formalism facilitates comparisons between predictions of quantum mechanics and the classical limit.

To solve the problem of the time-dependence induced by a perturbation V, it is very useful to define an intermediate picture called the *interaction picture.* For $H = H_0 + \lambda V$, let H_0 produce the time-dependence of operators and let the rest of the time-dependence be carried out on the states. Explicitly, we write

$$|\Psi_I(t)\rangle = e^{iH_0t}|\Psi(t)\rangle = e^{iH_0t}e^{-iHt}|\Psi(0)\rangle$$

$$\mathcal{O}_I(t) = e^{iH_0t}\,\mathcal{O}\,e^{-iH_0t}$$

The matrix elements of operators in states, by construction, have the same time-dependence as in the other pictures. The advantage of the interaction picture is that the transitions induced by the perturbation V are much more visible in the form of the states.

Let us then compute the equation of motion for the interaction picture wavefunction $|\Psi_I(t)\rangle$.

$$\begin{aligned} i\frac{\partial}{\partial t}|\Psi_I(t)\rangle &= e^{iH_0t}(-H_0)e^{-iHt}|\Psi\rangle \\ &\quad + e^{iH_0t}(H)e^{-iHt}|\Psi\rangle \\ &= e^{iH_0t}(H-H_0)e^{-iH_0t}e^{iH_0t}e^{-iHt}|\Psi\rangle \\ &= e^{iH_0t}\lambda V e^{-iH_0t}|\Psi_I(t)\rangle \end{aligned}$$

So, finally,

$$i\frac{\partial}{\partial t}|\Psi_I(t)\rangle = \lambda V_I(t)|\Psi_I(t)\rangle$$

where

$$\lambda \underset{\sim}{V_I}(t) = e^{iH_0t}\lambda \underset{\sim}{V} e^{-iH_0t}$$

is the perturbation with the time-dependence of the interaction picture.

It is straightforward to formally integrate the equation of motion for $|\Psi_I(t)\rangle$ in a power series in λ. Let the initial condition for the wavefunction be

$$|\Psi_I(t=0)\rangle = |\psi_0\rangle$$

Plugging this into the right-hand side of the equation, we find, to first order

$$|\Psi_I(t)\rangle = |\psi_0\rangle - i\int_0^t dt_1\, \lambda \underset{\sim}{V_I}(t_1)|\psi_0\rangle$$

Plugging this result back, we can work out the solution order by order. The result is

$$\begin{aligned}|\Psi_I(t)\rangle = |\psi_0\rangle &- i\int_0^t dt_1\, \lambda V_I(t_1)|\psi_0\rangle \\ &+ (-i)^2\int_0^t dt_1\int_0^{t_1} dt_2\, \lambda V_I(t_1)\lambda V_I(t_2)|\psi_0\rangle \\ &+ (-i)^3\int_0^t dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3\, \lambda V_I(t_1)\lambda V_I(t_2)\lambda V_I(t_3)|\psi_0\rangle + \cdots\end{aligned}$$

Note the ordering of the factors of $V_I(t)$ and of the limits of integration. The series is arranged so that factors of $V_I(t)$ evaluated at later times are put to the left of factors of $V_I(t)$ evaluated at earlier times. It is important to be precise about this, because in general the operators $V_I(t)$ at non-equal times will not commute. It is straightforward to verify this solution to the differential equation by differentiating it with respect to t and matching terms of the same order in λ.

The solution for $|\Psi_I(t)\rangle$ can be written more compactly by defining a *time-ordering symbol* T by

$$T[V_I(t_1)V_I(t_2)] = \begin{cases} V_I(t_1)\,V_I(t_2) & t_1 > t_2 \\ V_I(t_2)\,V_I(t_1) & t_1 < t_2 \end{cases}$$

Then the T symbol formally keeps track of the ordering of the factors $V_I(t)$. We can extend all of the time integrals to the domain $(0,t)$ and compensate this by dividing the term with n integrals by $n!$. Then

$$|\psi_I(t)\rangle = |\psi_0\rangle + (-i)\int_0^t dt_1 \; \lambda V_I(t_1)|\psi_0\rangle$$
$$+ \frac{(-i)^2}{2!}\int_0^t dt_1 \int_0^t dt_2 \; T[\lambda V_I(t_1)\lambda V_I(t_2)]|\psi_0\rangle$$
$$+ \frac{(-i)^3}{3!}\int_0^t dt_1 dt_2 dt_3 \; T[\lambda V_I(t_1)\lambda V_I(t_2)\lambda V_I(t_3)]|\psi_0\rangle + \ldots$$

This sums up to the form

$$|\psi_I(t)\rangle = T\left\{\exp\left[-i\int_0^t dt' \; \lambda V_I(t')\right]\right\}|\psi_0\rangle$$

However, please be careful with this expression. To evaluate it, you must expand the exponential in a power series and put the factors of $V_I(t)$ in each term into time order.

To begin with this formalism, I would like to work through a very simple example. Let $|a\rangle$ and $|b\rangle$ be states of an atom with Hamiltonian H_0,

$$H_0|a\rangle = E_a|a\rangle \qquad H_0|b\rangle = E_b|b\rangle \qquad E_b > E_a$$

If we turn on a static electric field $\vec{E}_0$, this can mix the two states. The potential due to the electric field is

$$V = -e\vec{E}_0\cdot\vec{r}$$

Let

$$v = \langle b|\, e\vec{E}_0\cdot\vec{r}\, |a\rangle$$

Then the lowest eigenstate of the Hamiltonian including the electric field will be

$$|a\rangle + \frac{v}{E_b - E_a} |b\rangle$$

If the electric field is weak, this will be only a small perturbation of the original state $|a\rangle$.

However, if we were to make the electric field time-dependent, it could more effectively drive transitions between $|a\rangle$ and $|b\rangle$. In particular, the energy difference $(E_b - E_a)$ is analogous to the frequency of an oscillator. If we perturb the system at a frequency ω that is close to this resonant frequency, we should be able to drive $|a\rangle$ into $|b\rangle$ with a high probability.

Consider, then, the Hamiltonian

$$H = H_0 + e\vec{E}_0 \cdot \vec{r} \cos \omega t$$

where the second term is considered as a perturbation. The interaction picture perturbation is

$$V_I(t) = e^{iH_0 t} \left(e\vec{E}_0 \cdot \vec{r} \cos \omega t \right) e^{-iH_0 t}$$

Notice that this operator has time-dependence from two sources, first, from the explicit time-dependence of the field, and, second, from the factors of $e^{iH_0 t}$.

Let the initial condition for the wavefunction be

$$|\psi_I(0)\rangle = |a\rangle$$

Then the amplitude for the system to be in $|b\rangle$ at time t is

$$-i\int_0^t dt_1 \quad \langle b| V_I(t)|a\rangle$$

It is straightforward to evaluate this and perform the integral

$$= -i\int_0^t dt_1 \quad e^{iE_b t_1} e^{-iE_a t_1} \quad \mathcal{V} \cos \omega t_1$$

$$= -\frac{i}{2}\int_0^t dt_1 \; [e^{i(E_b - E_a + \omega)t_1} + e^{i(E_b - E_a - \omega)t_1}]\cdot \mathcal{V}$$

$$= -\frac{i\mathcal{V}}{2}\left[\frac{(e^{i(E_b-E_a+\omega)t}-1)}{i(E_b-E_a+\omega)} + \frac{(e^{i(E_b-E_a-\omega)t}-1)}{i(E_b-E_a-\omega)}\right]$$

The two terms in the last line have very different magnitudes. The first term is small, of the order of $\mathcal{V}/(E_b - E_a)$, the same order as the static perturbation. The second term has a vanishing denominator of ω is close to $(E_b - E_a)$, and so it can be much larger. I will keep only this second term in the following discussion.

With this approximation, we see that

$$|\langle b| \Psi_I(t)\rangle|^2 = \frac{\mathcal{V}^2}{4} \; \frac{\sin^2\left(\frac{\omega - (E_b - E_a)}{2} t\right)}{\left(\frac{\omega - (E_b - E_a)}{2}\right)^2}$$

For $\omega = E_b - E_a$, this expression behaves as

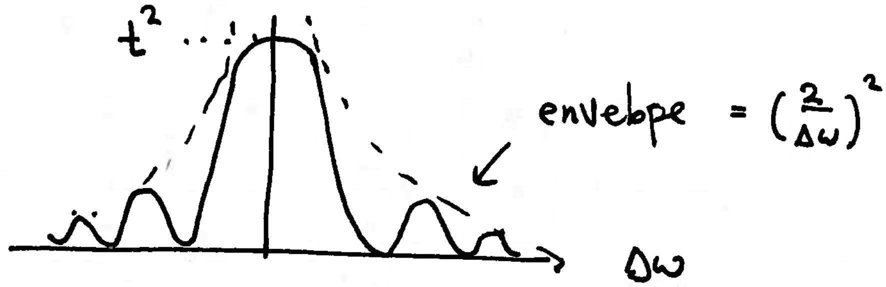
$$|\langle b| \Psi_I(t)\rangle|^2 \sim \frac{\mathcal{V}^2}{4} \cdot \; t^2$$

When this quantity becomes of order 1, the approximation of keeping only the first term in the perturbation series will break down. Nevertheless, if $\mathcal{V}$ is small, this term can be run up to very large times and can give a substantial probability for a transition from $|a\rangle$ to $|b\rangle$.

The functional that appears in this answer will be useful to us, so I would like to study it in more detail. Define

$$\delta_t(\Delta\omega) = \frac{\sin^2(\frac{1}{2}\Delta\omega\, t)}{(\frac{1}{2}\Delta\omega)^2}$$

This has the form



The function has height t^2 at $\Delta\omega = 0$. Its first zeros are at

$$\Delta\omega = \pm\frac{2\pi}{t}$$

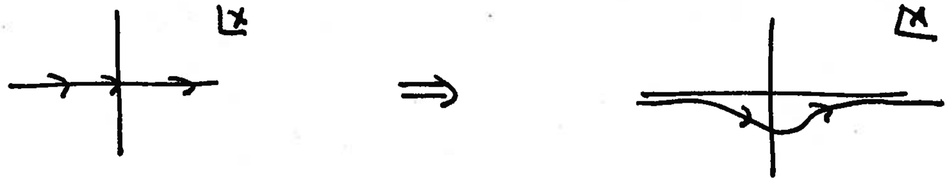
The second peaks, at $\pm 3\pi/t$, have a height of only

$$\frac{t^2}{(3\pi/2)^2} \sim 4.5\% \cdot t^2$$

The area under the function is

$$\int d(\Delta\omega)\, \delta_t(\Delta\omega) = 2t \int_{-\infty}^{\infty} dx\, \frac{\sin^2 x}{x^2}$$

This makes sense; the function has a height proportional to t^2 but a width proportional to $1/t$. The dimensionless integral is fun to evaluate using contour integration. The integrand has no singularity at $x = 0$, so we can push the contour downward.

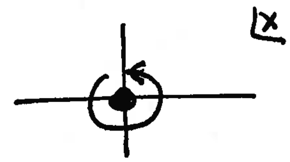


Now we can expand the function $\sin x$,

$$\int dx \, \frac{\sin^2 x}{x^2} = \int dx \left(-\frac{1}{4}\right) \frac{1}{x^2} \left(e^{2ix} - 2 + e^{-2ix}\right)$$

As we discussed in the previous lecture, we can evaluate a term with a negative exponential by pushing the contour down in the complex plane. Using this observation, we see that the third term, and also the second term, give zero. For the first term, we must push the contour upward in the complex plane and pick up the pole at $x = 0$.

$$-\frac{1}{4} \oint dx \, \frac{1}{x^2} \, e^{2ix}$$



Actually, there is a double pole at $x = 0$, so we must expand the numerator to find a single pole that will give a nonzero residue

$$-\frac{1}{4} \oint dx \, \frac{1}{x^2} \left(1 + 2ix + \cdots\right)$$

The value of the integral is then

$$(2\pi i)\left(-\frac{1}{4}\right)(2i) = \pi$$

that is

$$\int_{-\infty}^{\infty} dx \; \frac{\sin^2 x}{x^2} = \pi$$

We have now seen that the function we are studying has all of its support at $\Delta\omega < C/t$ and that

$$\int_{-\infty}^{\infty} d(\Delta\omega) \; \delta_t(\Delta\omega) = 2\pi t$$

As t becomes large, then, this function becomes a representation of the Dirac delta function

$$\delta_t(\Delta\omega) \longrightarrow 2\pi t \, \delta(\Delta\omega)$$

In the problem at hand, this limit applies when $t \gg 1/\omega$ or $t \gg (E_b - E_a)$. In the problem we were studying, there was only a single frequency ω. However, if we had a range of frequencies, we now see that we could trade a time dependence proportional to t^2 for a time-dependence proportional to t and an integration over a continuum of frequencies.

Let us now apply this logic to the analysis of the Wigner-Weisskopf problem. I would like to compute the transition rate from the isolated state $|0\rangle$ to the free-particle continuum using time-dependent perturbation theory.

To make this computation more well-defined, I would like to break up the continuum of free-particle states into discrete momentum eigenstates. Consider, then, putting the system in a large box of size L with periodic boundary conditions. The possible momenta then take the discrete values

$$\vec{k}_n = \frac{2\pi}{L} \vec{n} \qquad\qquad \vec{n} = (n_x, n_y, n_z)$$

The properly normalized wavefunctions of these states are

$$\langle \vec{r} \,|\, \vec{k}_n \rangle = \frac{e^{i \vec{k}_n \cdot \vec{r}}}{\sqrt{L^3}}$$

To begin, let us compute the amplitude as a function of time for the state $|0\rangle$ to make a transition to the discrete state $|k_n\rangle$. To first order in time-dependent perturbation theory, this is given by

$$\langle k_n | \Psi_I(t) \rangle = -i \int_0^t dt_1 \; \langle k_n | \, e^{i H_0 t_1} \lambda \chi(r) \, e^{-i H_0 t_1} \, | 0 \rangle$$

We can evaluate the exponentials and perform the integral

$$= -i \int_0^t dt_1 \; e^{i k^2/2m \, t_1} e^{-i E_0 t_1} \langle k_n | \lambda \chi(r) | 0 \rangle$$

$$= -i \frac{1}{i (k^2/2m - E_0)} \left(e^{i (k^2/2m - E_0) t} - 1 \right) \langle k_n | \lambda \chi(r) | 0 \rangle$$

The remaining matrix element can be evaluated by inserting a complete set of position eigenstates

$$\langle k_n | \lambda \chi(r) | 0 \rangle = \int d^3 r \; \langle k_n | r \rangle \langle r | \lambda \chi(r) | 0 \rangle$$

$$= \int d^3 r \; \frac{e^{-i \vec{k}_n \cdot \vec{r}}}{\sqrt{L^3}} \lambda \chi(r) = \frac{1}{\sqrt{L^3}} \lambda \tilde{\chi}(k_n)$$

We find, finally,

$$\langle k_n | \Psi_I(t) \rangle = - \frac{e^{i (k^2/2m - E_0) t} - 1}{(k^2/2m - E_0)} \frac{1}{\sqrt{L^3}} \lambda \tilde{\chi}(k_n)$$

To find the total probability for a transition, this result should be squared and summed over all possible momenta k_n.

$$\mathrm{Prob} = \sum_n |\langle k_n | \Psi_I(t) \rangle|^2$$

The sum over the closely spaced discrete states can be replaced by an integral

$$\sum_n = \int d^3n = \int d^3k \left(\frac{L}{2\pi}\right)^3$$

Then

$$\mathrm{Prob.} = \frac{L^3}{(2\pi)^3} \int d^3k \quad \delta_t\left(\frac{k^2}{2m} - \epsilon_0\right) \frac{1}{L^3} \quad \lambda^2 |\tilde{\chi}(k)|^2$$

Note that all reference to the size of the box has disappeared. For $t \gg E_0$, we can approximate the integrand by a delta function. Then the total probability of making a transition from $|0\rangle$ to the continuum is

$$\mathrm{Prob} = \int \frac{d^3k}{(2\pi)^3} \quad 2\pi t \, \delta\left(\frac{k^2}{2m} - \epsilon_0\right) \quad \lambda^2 \, |\tilde{\chi}(k)|^2$$

This result is proportional to t. That is, it predicts a uniform decay rate per unit time, leading to an exponential decay law. The rate of decay of the state $|0\rangle$ is

$$\Gamma = \int \frac{d^3k}{(2\pi)^3} \quad 2\pi \, \delta\left(\frac{k^2}{2m} - \epsilon_0\right) \quad \lambda^2 \, |\tilde{\chi}(k)|^2$$

It is interesting to perform the integral and check the consistency with our exact solution to the Wigner-Weisskopf model. The integral is spherically symmetric, so

$$\Gamma = \int \frac{dk\, k^2\, 4\pi}{8\pi^3}\; 2\pi\, \delta\!\left(\frac{k^2}{2m} - E_0\right) \lambda^2 |\tilde{\chi}(k)|^2$$

To integrate over the delta function, use the rule

$$\int dx\; \delta(f(x)) = \int df\, \left(\frac{dx}{df}\right) \cdot \delta(f) = \left(\frac{df}{dx}\right)^{-1}\Bigg|_{f=0}$$

Then

$$\Gamma = \frac{k^2}{\pi} \cdot \left(\frac{k}{m}\right)^{-1} \cdot \lambda^2 |\tilde{\chi}(k)|^2 \Bigg|_{k=(2mE_0)^{1/2}}$$

so that, finally

$$\Gamma = \frac{\lambda^2 m k}{\pi} |\tilde{\chi}(k)|^2 \Bigg|_{k=(2mE_0)^{1/2}}$$

This is exactly the result that we found from the exact solution, except for the shift of the resonance energy from E_0 to E_*, manifestly an effect of higher order in λ.

This new derivation of the expression for Γ does not rely on an exact solution. In fact, it is completely general, requiring only that λ is small enough that the first order perturbation theory formula is valid. Here is a more general way to state the result: Let $H = H_0 + \lambda V$. Let $|0\rangle$ be an eigenstate of H_0 that is made unstable by V and allowed to decay into a continuum free particle state. Let $|k\rangle$ be a *continuum-normalized* free particle state, that is

$$\langle \vec{r} | \vec{k} \rangle = e^{i\vec{k}\cdot\vec{r}}$$

Then the decay rate of $|0\rangle$ is given by

$$\Gamma = \int d\Pi \, |\langle k | \lambda V | 0 \rangle|^2$$

where

$$\int d\Pi = \int \frac{d^3k}{(2\pi)^3} \, 2\pi \, \delta(E(k) - E_0)$$

and $E(k)$ is the energy of the free particle. This latter integral is called the *integral over phase space.* For a nonrelativistic particle with energy $E(k) = k^2/2m$,

$$\int d\Pi = \frac{mk}{\pi} \cdot \int \frac{d\Omega}{4\pi} \Bigg|_{k=(2mE_0)^{1/2}}$$

However, the derivation I have given applies equally well whatever the energy-momentum relation. We can use the formula also for decay to a relativistic particle, or even to a photon. For a photon, $E(k) = ck$, and

$$\int d\Pi = \frac{k^2}{\pi c} \int \frac{d\Omega}{4\pi} \Bigg|_{k=E_0/c}$$

This master formula for Γ is called *Fermi's Golden Rule.*

Though this formula for Γ is amazingly general, it is interesting to try to extend it even further. I will now apply the same derivation to transitions from a continuum state to another continuum state. This would be a scattering process, in which a particle starts in a momentum state $|k\rangle$ and is driven by a perturbation λV to make a transition to a momentum state $|p\rangle$. For example, V could be a static potential localized near $\vec{r} = 0$.

To make the derivation, it will be clearest if we return to discrete states in a large box of size L with periodic boundary conditions and consider transitions from a discrete momentum state $|k_n\rangle$ to another discrete momentum state $|p_n\rangle$. I will take the initial condition for the wavefunction to be

$$|\Psi_I(t)\rangle = |k_n\rangle \quad \text{at} \quad t=0$$

and compute the probability for the transition using time-dependent perturbation theory. The amplitude for the transition is

$$-i\int_0^t dt_1 \; \langle p_n| \, e^{iH_0 t_1} \lambda V e^{-iH_0 t_1} |k_n\rangle$$

$$= -i\int_0^t dt_1 \; e^{i p^2/2m \, t_1} e^{-i k^2/2m \, t_1} \langle p_n|\lambda V|k_n\rangle$$

$$= -i \frac{1}{i(p^2/2m - k^2/2m)} \left(e^{i(p^2/2m - k^2/2m)t} - 1\right) \langle p_n|\lambda V|k_n\rangle$$

Squaring this expression, and changing from box normalization to continuum normalization, we find

$$|\langle p_n|\Psi_I(t)\rangle|^2 = \delta_t\left(\frac{p^2}{2m} - \frac{k^2}{2m}\right) \cdot \frac{|\langle p|\lambda V|k\rangle|^2}{L^3 \cdot L^3}$$

The sum over states $|p_n\rangle$ is given by

$$\sum_n = \int \frac{d^3p}{(2\pi)^3} \cdot L^3$$

Then the total probability to make a transition is

$$\sum_n |\langle p_n|\Psi_I(t)\rangle|^2 = \int \frac{d^3p}{(2\pi)^3} \, \delta_t\left(\frac{p^2}{2m} - \frac{k^2}{2m}\right) \frac{|\langle p|\lambda V|k\rangle|^2}{L^3}$$

For times $t \gg k^2/2m$,

$$\sum_n |\langle p_n | \Psi_I(t) \rangle|^2 = \int \frac{d^3p}{(2\pi)^3}\, 2\pi \cdot t \cdot \delta\left(\frac{p^2}{2m} - \frac{k^2}{2m}\right) \frac{|\langle p | \lambda V | k \rangle|^2}{L^3}$$

We can relate this result to the scattering cross section. Our calculation gives a definite rate for transitions per unit time.

$$\frac{\text{transitions}}{\text{sec}} = \frac{1}{L^3} \int \frac{d^3p}{(2\pi)^3}\, 2\pi\, \delta\left(\frac{p^2}{2m} - \frac{k^2}{2m}\right) |\langle p | \lambda V | k \rangle|^2$$

The scattering cross section is this rate divided by the flux

$$\Phi = \rho v = \frac{1}{L^3} v$$

Then the cross section

$$\sigma = \frac{\text{transitions/sec}}{\Phi}$$

is given by

$$\sigma = \frac{1}{v} \int d\Pi\; |\langle p | \lambda V | k \rangle|^2$$

where again we have an integral over phase space of the square of a matrix element. This is Fermi's Golden Rule for the scattering cross section. Again, the derivation make no reference to a specific form of the energy-momentum relation and is valid for nonrelativistic particles, relativistic particles, and photons. For photons, the velocity in the prefactor is of course the speed of light.

We can test this formula by evaluating it for the scattering of a nonrelativistic particle from a static potential. For this case

$$\langle \vec{p} | \lambda V | \vec{k} \rangle = \int d^3r \; \langle \vec{p} | \vec{r} \rangle \, \lambda V(r) \, \langle \vec{r} | \vec{k} \rangle$$

$$= \int d^3r \; e^{-i\vec{p}\cdot\vec{r}} e^{i\vec{k}\cdot\vec{r}} \lambda V(r) = \lambda \tilde{V}(p-k)$$

that is, the Fourier transform of the potential evaluated at the momentum transfer. Then, using the formula for the phase space integral for a nonrelativistic particle,

$$\sigma = \frac{1}{v} \frac{mk}{\pi} \int \frac{d\Omega}{4\pi} \left| \lambda \tilde{V}(p-k) \right|^2$$

$$= \frac{m}{k} \frac{mk}{\pi} \frac{1}{4\pi} \int d\Omega \left| \lambda \tilde{V}(p-k) \right|^2$$

or, finally,

$$\sigma = \int d\Omega \left| \frac{m}{2\pi} \left[\lambda \tilde{V}(p-k) \right] \right|^2$$

This agrees exactly with our earlier result for the Born approximation to the scattering cross section from a potential. This is what we should expect, since both calcuations are correct to order λ^2.

Fermi's Golden Rule can also be applied to situations in which the scattering amplitude is of higher order in λ. As an example, I will work out the leading order contribution to the scattering amplitude in the Wigner-Weisskopf problem. In that problem, the matrix elements of λV take a continuum state to the state $|0\rangle$ and vice versa, so we need to go to second order in time-dependent perturbation theory to find a transition amplitude from a discrete free particle state $|k_n\rangle$ to another discrete free particle state $|p_n\rangle$. The amplitude for such a transition in time t is

$$(-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \; \langle p_n | \lambda V_I(t_1) \, \lambda V_I(t_2) | k_n \rangle$$

To evaluate this, insert the intermediate state $|0\rangle\langle 0|$. Then we find

$$= (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \, \langle p_n| e^{iH_0 t_1} \lambda \chi(r) e^{-iH_0 t_1} |0\rangle\langle 0| e^{iH_0 t_2} \lambda \chi^*(r) e^{-iH_0 t_2} |k_n\rangle$$

$$= (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \, e^{i\frac{p^2}{2m}t_1} e^{-iE_0 t_1} \frac{\lambda \tilde{\chi}(p)}{L^{3/2}} e^{iE_0 t_2} e^{-i\frac{k^2}{2m}t_2} \frac{\lambda \chi^*(k)}{L^{3/2}}$$

$$= (-i)^2 \int_0^t dt_1 \, e^{i(\frac{p^2}{2m}-E_0)t_1} \lambda^2 \tilde{\chi}(p) \tilde{\chi}^*(k) \frac{e^{i(E_0 - k^2/2m)t_1} - 1}{i(E_0 - k^2/2m)}$$

$$= (-i)^2 \frac{1}{(-i)(k^2/2m - E_0)} \left(\frac{\lambda^2 \chi(p) \chi^*(k)}{L^3} \right) \left[\frac{e^{i(\frac{p^2}{2m} - \frac{k^2}{2m})t} - 1}{i(p^2/2m - k^2/2m)} - \frac{e^{i(p^2/2m - E_0)t}}{i(p^2/2m - E_0)} \right]$$

For a momentum $\vec{p}$ such that $p^2/2m \neq E_0$, the second term never becomes large or leads to a time-dependence growing with t. I will therefore neglect it. The square of the first term will give a delta function

$$2\pi t \, \delta\left(\frac{p^2}{2m} - \frac{k^2}{2m}\right)$$

for times $t \gg (p^2/2m)^{-1}$. This gives a uniform rate of scattering per unit time. Thus, we find for the rate of scattering from $|k_n\rangle$ to $|p_n\rangle$

$$\frac{1}{(k^2/2m - E_0)^2} \lambda^2 \frac{\chi(p) \chi^*(k)}{L^3 \cdot L^3} \cdot 2\pi t \, \delta\left(\frac{p^2}{2m} - \frac{k^2}{2m}\right)$$

Going back to the previous derivation for the cross section and repeating the remaining steps, we find in this case

$$\sigma = \frac{1}{v} \int d\Pi \left| \frac{\lambda^2 |\tilde{\chi}(k)|^2}{E(k) - E_0} \right|^2$$

We can now evaluate this using the expression for phase space for a nonrelativistic particle. That gives

$$\sigma = \frac{m}{k}\,\frac{mk}{\pi}\int\frac{d\Omega}{4\pi}\left|\frac{\lambda^2|\tilde{\chi}(k)|^2}{E(k)-E_0}\right|^2$$

or, finally,

$$\sigma = \int d\Omega\left|\frac{m}{2\pi}\,\lambda^2\frac{|\tilde{\chi}(k)|^2}{E(k)-E_0}\right|^2$$

This expression agrees with the leading order cross section derived using the Born approximation in the first lecture on the Wigner-Weisskopf problem.

Actually, with insight from the exact solution, we know that this formula must be modified for values of $\vec{k}$ near $E(k) = E_0$. The position of the resonance will be slightly shifted by higher-order corrections, and the resonance will be shifted into the complex plane by its decay. We can immediately write the final formula

$$\sigma = \int d\Omega\left|\frac{m}{2\pi}\,\lambda^2\,\frac{|\tilde{\chi}(k)|^2}{E(k)-E_{*}+i\,\Gamma/2}\right|^2$$

where Γ can be computed using Fermi's Golden Rule as described earlier in this lecture. We have now recovered all of the important features of the exact solution to the Wigner-Weisskopf problem using time-dependent perturbation theory.

Fermi's Golden Rule is very general and applies to a very wide range of problems involving scattering and decay. In the next few weeks, we will see applications of this formula to a broad range of quantum mechanics problems.