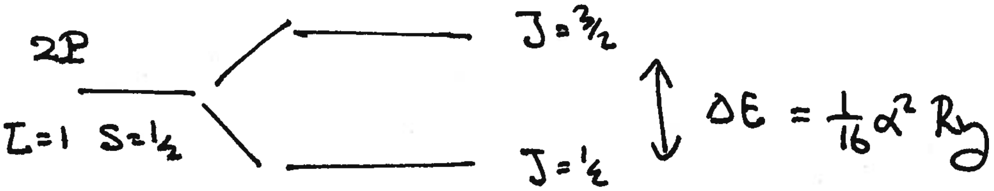
May 17

Electric Dipole Transitions

In this lecture, I will continue the study of photon absorption and emission through electric dipole transitions.

At the end of the previous lecture, we worked out the decay amplitudes for the decay of the 2P state of Hydrogen, ignoring the electron spin. Let's now put back the electron spin and see what changes.

It is not so obvious that the electron spin has any role to play in this problem. After all, the electric dipole matrix element does not involve the spin. However, the eigenstates of the Hamiltonian of the Hydrogen atom involve spin when we include the fine structure, which couples spin and orbital degrees of freedom. For the 2P states, the actual energy eigenstates are:



with a splitting of $\Delta E = \alpha^2\, Ry/16$.

I will now compute the decay rates of these states separately. Begin with the $2P_{1/2}$ state. For $j^3 = 1/2$, this is

$$\left|J=\tfrac{1}{2}\; j^3=\tfrac{1}{2}\right\rangle = \left\langle J=\tfrac{1}{2}\; j^3=\tfrac{1}{2}\right|\left. m=0\; s^3=\tfrac{1}{2}\right\rangle \left|m=0\; s^3=\tfrac{1}{2}\right\rangle$$

$$+ \left\langle J=\tfrac{1}{2}\; j^3=\tfrac{1}{2}\right|\left. m=1\; s^3=-\tfrac{1}{2}\right\rangle \left|m=1\; s^3=-\tfrac{1}{2}\right\rangle$$

$$= -\sqrt{\tfrac{1}{3}}\left|m=0\; s^3=\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|m=1\; s^3=-\tfrac{1}{2}\right\rangle$$

In the previous lecture, I computed the decay amplitudes for states of definite m. In the decay, the value of s^3 is unchanged. So, the state $\left|m=0,\, s^3=\tfrac{1}{2}\right\rangle$ will decay to $\left|1S,\, s^3=\tfrac{1}{2}\right\rangle$ at the rate computed there,

$$\Gamma(2P) \equiv \frac{2^9}{3^8}\,\alpha^3 \,\mathrm{Ry}$$

and the state $\left|m=1,\ s^3=-\tfrac{1}{2}\right\rangle$ will decay to $\left|1\mathrm{S},\ s^3=-\tfrac{1}{2}\right\rangle$ at the same rate. The final states are distinct, differing in the s^3 quantum number, so we should square the two decay amplitudes separately and add the rates. The total decay rate is then

$$\Gamma(2P_{1/2}) = \frac{1}{3}\Gamma(2P) + \frac{2}{3}\Gamma(2P) = \Gamma(2P)$$

and is unchanged from the earlier discussion. However, the angular distribution of the photons is changed,

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta} = \frac{1}{3}\left(\frac{3}{4}\sin^2\theta\right) + \frac{2}{3}\left(\frac{3}{8}(1+\cos^2\theta)\right)$$

$$= \frac{1}{4}\sin^2\theta + \frac{1}{4} + \frac{1}{4}\cos^2\theta = \frac{1}{2}$$

The distribution of the photons emitted in the decay of the $2\mathrm{P}_{1/2}$ state is *isotropic*.

We can analyze the $2\mathrm{P}_{3/2}$ states similarly. These states are

$$\left|J=\tfrac{3}{2}\ j^3=\tfrac{3}{2}\right\rangle = \left|m=1\ s^3=\tfrac{1}{2}\right\rangle$$

$$\left|J=\tfrac{3}{2}\ j^3=\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|m=0\ s^3=\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|m=1\ s^3=-\tfrac{1}{2}\right\rangle$$

By the same logic, the total decay rates are the same. For the $j^3 = 3/2$ state, the photon angular distribution is

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta} = \frac{3}{8}(1+\cos^2\theta)$$

and for the $j^3 = 1/2$ state, the photon angular distribution is

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta} = \frac{2}{3}\left(\frac{3}{4}\sin^2\theta\right) + \frac{1}{3}\left(\frac{3}{8}(1+\cos^2\theta)\right)$$

$$= \frac{5}{8} - \frac{3}{8}\cos^2\theta$$

A check on these results is that a random mixture of the four $J = 3/2$ states should give an isotropic angular distribution, and it does.

What is odder is the result for the experiment described at the end of the previous lecture. We shine a beam of light along the $\hat{3}$ axis onto a sample of Hydrogen atoms.

The angular distribution of the emitted photons depends on the energy of the photons, that is, on which fine structure level they excite. The results are

$J = \frac{1}{2}$ isotropic

$J = \frac{3}{2}$ $\frac{7}{16} + \frac{3}{16}\cos^2\theta$

It is strange that, first, these results are both different from the results found at the end of the previous lecture, and that, second, both results are closer to isotropic. What is going on?

To explore this, I would like to write out the photon scattering cross section, following our discussion of Fermi's Golden Rule for resonance excitation. We found that the cross section for scattering through a resonant state is given by a formula

$$\sigma(a+\gamma\to i\to a+\gamma) = \frac{1}{V}\int d\Pi \left|\frac{\langle a\gamma|\Delta H|i\rangle \quad \langle i|\Delta H|a\gamma\rangle}{E_i - E_a - \omega + i\Gamma/2}\right|^2$$

in which the squared amplitude contains a matrix element for the production of the resonance, a matrix element for the decay of the resonance, and a denominator that gives the Lorentzian line shape. I have added the width of the resonance in the way that we know is appropriate from the Wigner-Weisskopf problem. Fermi's Golden Rule gives the width as

$$\Gamma = \int d\Pi \; |\langle a\gamma|\Delta H|i\rangle|^2$$

To understand this formula a little better, write it as

$$\sigma(a+\gamma \to i \to a+\gamma) = \frac{1}{V} \int d\Pi \, |\langle a\gamma|\Delta H|i\rangle|^2 \, \frac{|\langle i|\Delta H|a\gamma\rangle|^2}{(E_i - E_a - \omega)^2 + (\Gamma/2)^2}$$

We can recognize that this falls into the form

$$\sigma(a+\gamma \to i \to a+\gamma) = \frac{2}{V} \left(\frac{\Gamma/2}{(E_i - E_a - \omega)^2 + (\Gamma/2)^2} \right) |\langle i|\Delta H|a\gamma\rangle|^2$$

Now consider the limit of a very narrow width. The function of ω is strongly peaked about the value

$$\omega = E_i - E_a$$

and the integral over this function is

$$\int_{-\infty}^{\infty} d\omega \, \frac{\Gamma/2}{(\omega - (E_i - E_a))^2 + (\Gamma/2)^2} = \tan^{-1} \frac{\omega - (E_i - E_a)}{\Gamma/2} \Bigg|_{-\infty}^{\infty} = \pi$$

Thus, as $\Gamma \to 0$, the function tends to

$$\frac{\Gamma}{(E_i - \epsilon_a - \omega)^2 + (\Gamma/2)^2} \xrightarrow[\Gamma \to 0]{} 2\pi\, \delta(E_i - \epsilon_a - \omega)$$

and our expression for the cross section becomes

$$\sigma(a+\gamma \to i) \longrightarrow \frac{1}{V}\, 2\pi\, \delta(E_i - \epsilon_a - \omega)\, \left| \langle i | \Delta H | a\gamma \rangle \right|^2$$

This is just the Fermi's Golden Rule expression for production of a long-lived resonance.

Notice that the two cross sections that we are discussing here are actually for different processes, on the one hand, the scattering of a photon with a resonance in the intermediate state, on the other hand, the absorption of a photon to produce the resonance. However, if Γ → 0, the resonance is very long-lived and we can consider it as a stable particle. Conversely, if $\Gamma \neq 0$, the resonance is unstable. Then it is ephemeral, and the only final state present after a long time will be a decay product of the resonance.

Next, what if there are several resonances that contribute to the scattering process? If we observe the final photon but do not disturb the system during the scattering process, we must add the effects of the resonances *in the amplitude.* This gives

$$\sigma(a+\gamma \to a+\gamma) = \frac{1}{V} \int d\Pi \left| \sum_i \frac{\langle a\gamma | \Delta H | i \rangle \langle i | \Delta H | a\gamma \rangle}{E_i - \epsilon_a - \omega + i\Gamma_i/2} \right|^2$$

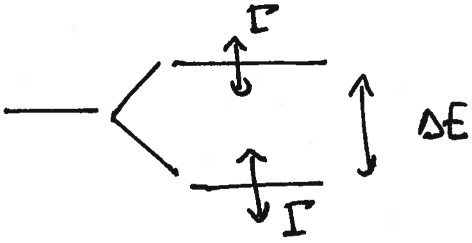
An alternative way to derive this expression is to go back to the derivation of the cross section formula using second-order time-dependent perturbation theory and evaluate the expression we found there by inserting a complete set of intermediate states

$$\langle a\gamma | \Delta H_I(t_1)\, \Delta H_I(t_2) | a\gamma \rangle$$

$$= \sum_i \langle a\gamma | \Delta H_I(t_1) | i \rangle \langle i | \Delta H_I(t_2) | a\gamma \rangle$$

In any event, the correct prescription in quantum mechanics is to compute the amplitude as a sum over resonances and then square the result.

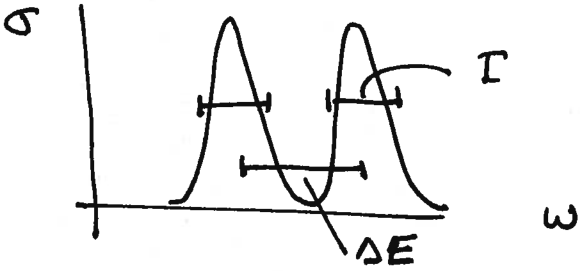
The physics of this expression is especially interesting in the case where we have two closely space resonances, with energy spacing ΔE and width Γ. (I will take the two widths to be equal for simplicity.)



There are two extreme cases with different physics. One limit is

$$\Delta E \gg \Gamma$$

In this case, the two resonances do not overlap in energy. The cross section as a function of energy has the form



In the vicinity of each resonance, we can ignore the other one. The cross section is well approximated by the expression

$$\sigma \approx \sum_i \frac{1}{V} \int d\Pi \left| \frac{\langle a\gamma|\Delta H|i\rangle\langle i|\Delta H|a\gamma\rangle}{E_i - E_a - \omega + i\Gamma/2} \right|^2$$

We say that the resonances contribute *incoherently* to the cross section. This is the treatment that we used to compute photon angular distributions for scattering through the $J = 1/2$ and $J = 3/2$ 2P states at the beginning of this lecture.

On the other hand, if

$$\Delta E \ll \Gamma$$

the resonances overlap. It is a good approximation to factor the denominator out of the matrix element, giving the cross section in the form

$$\sigma \approx \frac{1}{V}\int d\Pi \, \frac{1}{(E_i - E_a - \omega)^2 + (\Gamma/2)^2} \left| \langle a\gamma | \, \Delta H \sum_i |i\rangle\langle i| \, \Delta H \, | a\gamma\rangle \right|^2$$

In this case, we say that the resonances contribute *coherently* to the cross section. For the case of the $J = 1/2$ and $J = 3/2$ 2P states, we can rearrange the sum over intermediate states

$$\sum_{J J^3} |J J^3\rangle\langle J J^3| = \sum_{m s^3} |m s^3\rangle\langle m s^3|$$

$$= \left(\sum_m |m\rangle\langle m|\right) \cdot \left(\sum_{s^3} |s^3\rangle\langle s^3|\right)$$

Now the electron spin factors out and plays no role. This gives the prediction for the photon angular distribution found at the end of the previous lecture.

In the actual case of 2P → 1S + γ transitions, the correct limit is

$$\Delta E \gg \Gamma$$

so the predictions given in this lecture are correct and those in the previous lecture are wrong. However, to conclude this, it was necessary to explicitly analyze the relation between ΔE and Γ. This relation is different for every system.

There is a nice physical explanation of these results, involving the reciprocal relation between energy and time. The time

$$\hbar/\Delta E$$

is the time it takes to for the atom to oscillate between the states

$$|m=0\ s^3=\tfrac{1}{2}\rangle \rightleftharpoons |m=1\ s^3=-\tfrac{1}{2}\rangle$$

and thus to establish the fine structure energy eigenstates. The time

$$\hbar/\Gamma$$

is the decay time. If

$$\Gamma \gg \Delta E$$

then the resonant state decays before angular momentum can be exchanged between the orbital and spin degrees of freedom. In this case, the electron spin is irrelevant. On the other hand, if

$$\Delta E \gg \Gamma$$

then the angular momentum can equilibrate between the orbital motion and the spin in a time short compared to the decay time. Then the efffect of electron spin cannot be ignored, and it is also intuitive that the effect of electron spin will be to wash out the photon angular distribution into a more isotropic form.

After this concrete example, I would like to write some more general formulae for electric dipole transitions. In the detailed calculation I have just completed, we took account of each specific polarization states of the atoms. More often, we ignore the atomic and also the photon polarization. To compute an *unpolarized* cross section, that is, a cross section that ignores polarization, we *sum* over the spin degrees of freedom in the final state and *average* over the spin degrees of freedom in the initial state.

The matrix element for an electric dipole transition is

$$(2\pi\alpha\omega c)^{\frac{1}{2}} \langle b | \vec{\varepsilon} \cdot \vec{r} \, | a \rangle$$

Treating this more generally, $|a\rangle$ is a member of an angular momentum multiplet with angular momentum J_a and $|b\rangle$ is a member of an angular momentum multiplet with angular momentum J_b. To compute the unpolarized cross section for

$$\sigma(a+\gamma \to b)$$

we sum over the $(2J_b + 1)$ states of the b multiplet, and we average over the $(2J_a + 1)$ states of the a multiplet and over the 2 photon polarizations.

It is interesting to express the cross section in terms of the squared matrix element with all polarizations summed over,

$$\Xi = \sum_{j^3(a)\, j^3(b)\, k} \left| \langle b | r^k | a \rangle \right|^2$$

This expression involves the sum

$$\sum_{j^3(a)\, j^3(b)} \langle a | r^i | b \rangle \langle b | r^j | a \rangle$$

summing over all rotational states of a and b. This object has no intrisic orientation, and so it must be proportional to δ^{ij}. Then

$$\sum_{j^3(a),\, j^3(b)} \langle a|r^i|b\rangle\langle b|r^j|a\rangle = \frac{1}{3}\delta^{ij}\sum_{j^3(a),\, j^3(b),\, k}\langle a|r^k|b\rangle\langle b|r^k|a\rangle$$
$$= \frac{1}{3}\delta^{ij}\,\Xi$$

and so

$$\sum_{j^3(a),\, j^3(b)} \langle a|\vec{\varepsilon}^{\,*}\cdot\vec{r}|b\rangle\langle b|\vec{\varepsilon}\cdot\vec{r}|a\rangle = \frac{1}{3}\vec{\epsilon}^{\,*}\cdot\vec{\epsilon}\;\Xi = \frac{1}{3}\Xi$$

Finally, we find

$$\sum_{j^3(a),\, j^3(b),\, \varepsilon} \left|\langle b|\vec{\varepsilon}\cdot\vec{r}|a\rangle\right|^2 = \frac{2}{3}\Xi$$

The unpolarized cross section for $a + \gamma \to b$ can then be written

$$\sigma(\gamma+a\to b)\Big|_{\text{unpol.}} = \frac{1}{c}\,2\pi\delta(E_b - E_a - \omega)\;2\pi\alpha\omega c\cdot\frac{1}{2(2J_a+1)}\sum_{\substack{j^3(a)\, j^3(b)\\ \varepsilon}}\left|\langle b|\vec{\varepsilon}\cdot\vec{r}|a\rangle\right|^2$$

or, more simply

$$\sigma(\gamma+a\to b)\Big|_{\text{unpl.}} = \frac{4}{3}\pi\alpha\omega\cdot 2\pi\delta(E_b + E_a - \omega)\cdot\frac{\Xi}{2(2J_a+1)}$$

We can apply the same logic to the calculation of the decay rate of the states $|b\rangle$. All of the states in the b multiplet have the same decay rate. We then average over the states of b and sum over the states of a and over the polarization of the photon. This gives

$$\Gamma(b\to a\gamma) = \frac{1}{2J_b+1}\int d\Pi\,(2\pi\alpha\omega c)\sum_{j^3(a)j^3(b)\varepsilon}\left|\langle a|\vec{\varepsilon}^{\,*}\cdot\vec{r}\,|b\rangle\right|^2$$

$$= \frac{1}{2J_b+1}\,\frac{k^2}{\pi c}\,2\pi\alpha\omega c\cdot\frac{2}{3}\,\boxed{\;}$$

or, finally,

$$\Gamma(b\to a\gamma) = \frac{4\pi\alpha\omega}{3}\cdot\frac{k^2}{\pi}\cdot\frac{\boxed{\;}}{2J_b+1}$$

We can combine these formulae into the relation

$$\sigma(a+\gamma\to b) = \frac{\pi}{k^2}\,\frac{2J_b+1}{2(2J_a+1)}\,\Gamma(b\to a\gamma)\cdot 2\pi\delta(E_b-E_a-\omega)$$

This relation is impressively general. Actually, it does not depend on the structure of the electric dipole matrix element or on any detail of the resonance production and decay mechanism. It involves only the number of initial and final states and, in the prefactor, the form of phase space for photon emission. Similar formulae can be derived that relate any process of resonance production to the corresponding inverse process of resonance decay.

The factor

$$\sum_{j^3(b)\,k}\left|\langle a|r^k|b\rangle\right|^2$$

has another interesting property. Recall that

$$\langle b|\frac{p^k}{m}|a\rangle = \frac{1}{i}\langle b|\,[r^k, H_0]\,|a\rangle = \frac{E_a-E_b}{i}\langle b|r^k|a\rangle$$

$$\langle a|\frac{p^k}{m}|b\rangle = \frac{1}{i}\langle a|\,[r^k, H_0]\,|b\rangle = \frac{E_b-E_a}{i}\langle a|r^k|b\rangle$$

Then the sum over all states $|b\rangle$ can be written

$$\sum_{bk} (E_b - E_a)\ \langle a|r^k|b\rangle\langle b|r^k|a\rangle$$

$$= \sum_{bk} \frac{i}{2}\left[\langle a|\frac{p^k}{m}|b\rangle\langle b|r^k|a\rangle - \langle a|r^k|b\rangle\langle b|\frac{p^k}{m}|a\rangle\right]$$

$$= \sum_{k} \frac{i}{2m}\langle a|\,p^k r^k - r^k p^k\,|a\rangle$$

$$= \sum_{k} \frac{-i}{2m}\langle a|\,[r^k, p^k]\,|a\rangle = \sum_{k} \frac{1}{2m}\delta^{kk}\langle a|a\rangle$$

so that

$$\sum_{bk} \omega\,\left|\langle a|r^k|b\rangle\right|^2 = \frac{3}{2m}$$

Looking back at our expression for the unpolarized cross section, we see that we have computed the sum of this cross section over all states $|b\rangle$,

$$\int_0^\infty d\omega \sum_b \sigma(\gamma + a \to b)\Big|_{\text{unpol.}} = \frac{2\pi\alpha}{3}\sum_{bk}\int d\omega\ 2\pi\,\delta(E_b - E_a - \omega) \cdot \omega\,\left|\langle a|r^k|b\rangle\right|^2$$

In other words,

$$\int_0^\infty dE_\gamma\ \sigma(\gamma + a) = \frac{2\pi^2\alpha}{m}$$

This relation is called the *Thomas-Reiche-Kühn* or the *oscillator strength* sum rule. Supplying a factor $\hbar^2$ to obtain the correct dimensions and using

$$a_0 = \left(\frac{e^2}{4\pi\epsilon_0}\right)^{-1}\frac{\hbar^2}{m} = \frac{1}{\alpha}\,\frac{\hbar}{mc} \qquad R_y = \frac{1}{2}\,\frac{\alpha\hbar c}{a_0}$$

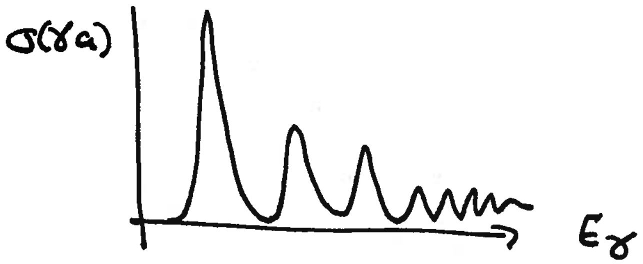
so that

$$2\pi^2\alpha \frac{\hbar^2}{m} = 2\pi^2 \alpha \hbar c \; a_0$$

we find the following expression for this sum rule

$$\int_0^\infty dE_\gamma \; \sigma(\gamma + a) = 4\pi^2 \cdot R_y \cdot a_0^2$$

If we picture the γa cross section as a sum over a series of resonances



the Thomas-Reiche-Kühn sum rule tells us that the total area under the curve is given from first principles, independently of the details of the atomic Hamiltonian. If an atom is a strong photon absorber, most of the area will be concentrated in a few large peaks at low values of the photon energy. If an atom is a weak photon absorber, the area under the curve is the same, but it is distributed over a very broad range in energy.

To the extent that the resonances can be treated incoherently, the sum over resonances is a sum of positive contributions

$$\sum_k (E_b - E_a) \left| \langle b | r^k | a \rangle \right|^2$$

Then, the contribution from any single resonance is bounded above by $(3/2m)$. That is

$$\sum_{j^3(a),\, j^3(b),\, k} (E_b - E_a) \left| \langle b | r^k | a \rangle \right|^2 \leq \frac{3}{2m} (2J_a + 1)$$

This bound implies

$$\Gamma(b \to a\gamma) \leq \frac{4\pi\alpha}{3} \cdot \frac{k^2}{\pi} \; \frac{2J_a+1}{2J_b+1} \; \frac{3}{2m}$$

which can be assembled into

$$\Gamma(b \to a\gamma) \leq 2\alpha \left(\frac{2J_a+1}{2J_b+1} \right) \frac{E_\gamma^2}{mc^2} = \alpha^3 \left(\frac{2J_a+1}{2J_b+1} \right) \frac{E_\gamma^2}{Ry}$$

When we computed the width of the 2P states explicitly, we found

$$\Gamma(2P) = \frac{2^9}{3^8} \alpha^3 Ry$$

Using $E_\gamma = \frac{3}{4} Ry$, this becomes

$$\Gamma(2P) = \frac{2^{13}}{3^{10}} \alpha^3 \frac{E_\gamma^2}{Ry} = 0.14 \, \alpha^3 \frac{E_\gamma^2}{Ry}$$

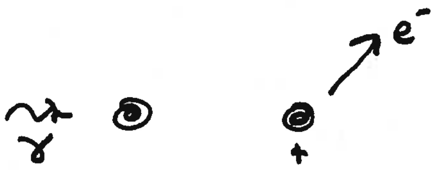
There are 6 2P states, decaying to 2 1S states. Then we should multiply this result by 6 and compare to the bound

$$6 \cdot \Gamma(2P \to 1S + \gamma) \leq 2 \, \alpha^3 \frac{E_\gamma^2}{Ry}$$

Then the 2P states account for 42% of the total oscillator strength in the $1S + \gamma$ reaction channel.

The integral in the Thomas-Reiche-Kühn sum rule extends over all energies, including energies sufficiently high that the electron is no longer bound to the atom. Thus, we are invited to think about processes in which an energetic photon kicks an electron out of an atom into the continuum. This is the *photoelectric effect.*

$$\gamma + (\text{atom}) \to (\text{atom})^+ + e^-$$



The first result in the quantum theory of atoms was Einstein's idea that quanta of light transfer just this quantum of energy to an atom and can only eject an electron if their energy is higher than the binding energy of the electron

$$E_\gamma = \hbar\omega > E_b$$

Now we have come back to this territory with all of the tools to calculate the cross section for the photoelectric effect.

From the Fermi Golden Rule,

$$\sigma = \frac{1}{v}\int d\Pi \; |\langle e^-(\vec{p})|\Delta H|a\gamma\rangle|^2$$

We should take $v = c$ for the incident photon and use the nonrelativistic phase space formulae for the final-state electron. For the matrix element, we can take the electric dipole matrix element

$$\langle e^-(p)|\Delta H|a\gamma\rangle = (2\pi\alpha\omega c)^{\frac{1}{2}} \langle \vec{p}\,|\,\vec{\epsilon}\cdot\vec{r}\,|a\rangle$$

Taking the simplest example, I will compute the photoelectric cross section on the 1S state of Hydrogen. Then the spin of the electron carries through simply from the initial state to the final state and we can ignore the spin in the cross section calculation. With these approximations, we find

$$\sigma = \frac{1}{c}(2\pi\alpha\omega c)\,\frac{mp}{\pi}\int\frac{d\Omega}{4\pi}\,\left|\langle\vec{p}\,|\,\vec{\epsilon}\cdot\vec{r}\,|\,1s\rangle\right|^2$$

where, following Einstein,

$$\frac{p^2}{2m} = \hbar\omega - Ry$$

Properly, we should take the final state $|\vec{p}\rangle$ to be an exact continuum eigenstate of the Hydrogen atom problem. However, except close to $E_\gamma = Ry$, it is not a bad approximation to take $|\vec{p}\rangle$ to be a free-particle plane wave state. Then

$$\langle\vec{p}\,|\,\vec{\epsilon}\cdot\vec{r}\,|\,1s\rangle = \int d^3r\; e^{-i\vec{p}\cdot\vec{r}}\;\vec{\epsilon}\cdot\vec{r}\;\frac{1}{\sqrt{\pi a_0^3}}\,e^{-r/a_0}$$

$$= \frac{1}{\sqrt{\pi a_0^3}}\; i\,\vec{\epsilon}\cdot\frac{\partial}{\partial\vec{p}}\int d^3r\; e^{-i\vec{p}\cdot\vec{r}}\,e^{-r/a_0}$$

The integral in this expression is

$$\int d^3r\; e^{-i\vec{p}\cdot\vec{r}}\,e^{-r/a_0} = 2\pi\int_0^\infty dr\,r^2\,\frac{1}{-ipr}\left(e^{-ipr} - e^{ipr}\right)e^{-r/a_0}$$

Since

$$\int_0^\infty dr\; r\,e^{-\lambda r} = \lambda^{-2}$$

we can evaluate the integral to find

$$\frac{2\pi i}{p}\left[\frac{1}{(1/a_0+ip)^2}-\frac{1}{(1/a_0-ip)^2}\right]$$

$$=\frac{2\pi}{p}\cdot 2\cdot\frac{2p/a_0}{[(1/a_0)^2+p^2]^2}=8\pi\frac{a_0^3}{(1+a_0^2p^2)^2}$$

Then

$$\langle\vec{p}|\vec{\epsilon}\cdot\vec{r}\,|1s\rangle=\frac{1}{\sqrt{\pi a_0^3}}\,i\vec{\epsilon}\cdot\frac{8\pi\cdot 4\vec{p}\,a_0^5}{(1+a_0^2p^2)^3}$$

Squaring this matrix element and assembling the pieces,

$$\sigma=2\pi\alpha\omega\frac{mp}{\pi}\int\frac{d\Omega}{4\pi}\frac{64\pi^2}{\pi a_0^3}\cdot 16\frac{a_0^{10}}{[1+(a_0p)^2]^6}(\vec{\epsilon}\cdot\vec{p})^2$$

$$=2^{11}\pi\alpha mp\omega\frac{a_0^7}{[1+(a_0p)^2]^6}\int\frac{d\Omega}{4\pi}(\vec{\epsilon}\cdot\vec{p})^2$$

Noticee that the direction in which the electron is ejected is guided by the photon polarization vector. For an unpolarized photon beam coming in along the $\hat{3}$ axis, the polarization average is computed using

$$\langle\epsilon^i\epsilon^{j*}\rangle=\frac{1}{2}\sum_{\epsilon}\epsilon^i\epsilon^{j*}=\frac{1}{2}\left(\delta^{ij}-\hat{3}^i\hat{3}^j\right)$$

to give

$$\langle|\vec{\epsilon}\cdot\vec{p}|^2\rangle=\frac{1}{2}(p^2-p^2\cos^2\theta)=\frac{1}{2}p^2\sin^2\theta$$

Then,

$$\sigma = \frac{2^{10}\pi\alpha m p^3 \omega a_0^7}{[1+(a_0 p)^2]^6}\int_{-1}^{1}\frac{d\cos\theta}{2}\sin^2\theta$$

$$= \frac{2^{11}}{3}\pi\alpha m p^3 \omega a_0^7/(1+(a_0 p)^2)^6$$

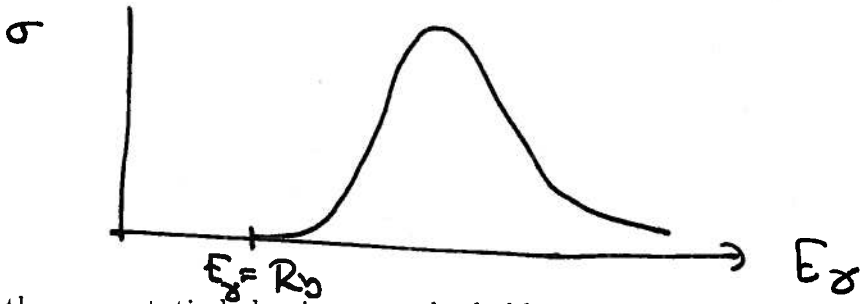
To obtain the right dimensions, add factors of $\hbar$ and employ the relations

$$m = \frac{Ry}{\frac{\alpha^2}{2}c^2} \qquad a_0 = \frac{\alpha\hbar c}{2Ry}$$

Then the final result is

$$\sigma(\gamma + H \to H^+ + e^-) = \frac{1024}{3}\pi\alpha\left(\frac{pa_0}{\hbar}\right)^3\frac{\hbar\omega}{Ry}\frac{a_0^2}{\left[1+\left(\frac{pa_0}{\hbar}\right)^2\right]^6}$$

The cross section has the form



with the asymptotic behavior near theshold

$$\sigma \sim (E_\gamma - Ry)^3$$

and the behavior at high energies $E_\gamma \gg Ry$.

$$\sigma \sim E_\gamma^{-7/2}$$