April 10

The Born Approximation

In the previous lecture, I explained that the cross section for scattering from a potential in 3 dimensions can be computed by solving the time-dependent Schrödinger equation, looking for a solution with the outgoing boundary conditions

$$\varphi_k(x) \sim e^{ikz} + \frac{f_k(\theta,\phi)}{r} e^{ikr}$$

for $r = |\vec{x}|$ much greater than the range of the potential. I showed that the coefficient in this asymptotic form, which I called the scattering amplitude, gives directly the differential scattering cross section through the formula

$$\frac{d^2\sigma}{d^2\Omega} = |f_k(\theta,\phi)|^2$$

In the next few lectures, I will give some methods for solving the Schrödinger equation to compute $f_k(\theta, \phi)$.

At the end of the previous lecture, I gave a formal solution for the Schrödinger wavefunction of a scattering problem as a perturbation series in the potential V. In this lecture, I will show how to use that series to make an approximate computation of $f_k(\theta, \phi)$.

Before we begin, though, we have one piece of unfinished business from the previous lecture. I claimed that the Green's function solving the equation

$$(-\nabla_x^2 - k^2) G(x,y) = \delta^{(3)}(\vec{x}-\vec{y})$$

was

$$G(x,y) = \frac{1}{4\pi r} e^{ikr} \qquad r = |\vec{x} - \vec{y}|$$

I would now like to prove that assertion.

My proof will be somewhat indirect, but it has other useful implications. I will start by consider the *Yukawa potential*

$$V(r) = \frac{g^2}{4\pi r} e^{-ar}$$

Yukawa showed that this is the form of the interaction potential associated with a force transmitted by a massive particle. The relation between the mass m of the particle and the range $1/a$ of the force is

$$\frac{1}{a} = \frac{\hbar}{mc}$$

You will notice that, when $m \to 0$, this potential becomes just the Coulomb potential. Yukawa applied this formula to build a theory of the nuclear form based on the exchange of the pi meson, which he postulated for this purpose.

I claim that the Fourier transform of the Yukawa potential is

$$\tilde{V}(\vec{p}) = \int d^3x \, e^{-i\vec{p}\cdot\vec{x}} V(\vec{x}) = \frac{g^2}{a^2 + p^2}$$

The proof is not difficult; we just need to do the integral. Going to polar coordinates,

$$\tilde{V}(\vec{p}) = \int_0^\infty dr \, r^2 \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \; e^{-ipr\cos\theta} \; V(r)$$

$$\tilde{V}(\vec{p}) = \int_0^\infty dr\, r^2 \quad \frac{1}{-ipr}\left(e^{-ipr} - e^{ipr}\right) \cdot 2\pi \cdot \frac{g^2}{4\pi r} e^{-ar}$$

$$= \frac{g^2}{2\cdot(-ip)} \int_0^\infty dr \; e^{-ar} \left(e^{-ipr} - e^{ipr}\right)$$

$$= \frac{g^2}{-2ip}\left(\frac{1}{a+ip} - \frac{1}{a-ip}\right) = \frac{g^2}{a^2+p^2}$$

Inverting the Fourier transform, we see that

$$\int \frac{d^3p}{(2\pi)^3} \; e^{i\vec{p}\cdot\vec{x}} \; \frac{g^2}{p^2+a^2} = \frac{g^2}{4\pi r} \; e^{-ar}$$

This integral is also not so hard to do directly, by contour integration.

Now we can solve the Green's function equation. Introduce the Fourier representation

$$G(x,y) = \int \frac{d^3p}{(2\pi)^3} \; e^{i\vec{p}\cdot\vec{x}} \; \tilde{G}(\vec{p})$$

and represent the delta-function by

$$\delta(\vec{x}-\vec{y}) = \int \frac{d^3p}{(2\pi)^3} \; e^{i\vec{p}\cdot\vec{x}} \cdot 1$$

Then the Green's function equation becomes

$$\int \frac{d^3p}{(2\pi)^3} \; e^{i\vec{p}\cdot\vec{x}} \; \left[(p^2-k^2)\, \tilde{G}(\vec{p}) - 1 \right] = 0$$

For this to be zero, the integrand must vanish; thus,

$$\tilde{G}(p) = \frac{1}{p^2 - k^2}$$

and the real space Green's fuction is

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 - k^2}$$

We recognize the Fourier integral as just the analytic continuation of the Yukawa potential by $a^2 \to -k^2$. There is one subtlety; the factor a^2 has been turned into a negative number, and we have to take the square root. In taking the square root of $(-k^2)$, we could make either of the choices $\pm ik$. The choice of $-ik$ gives the correct outgoing boundary conditions. Then, indeed,

$$G(x,y) = \frac{1}{4\pi r} e^{ikr} \qquad r = |\vec{x} - \vec{y}|$$

Now that I have completed the derivation of the perturbation series, I would like to give another, more streamlined, derivation. In quantum mechanics, we use a suggestive notation in which operators are viewed as matrices and the states they act on are written as bras and kets. This makes the derivation given in the previous lecture much simpler.

The original time-independent Schrödinger equation that we solved for $\varphi_k(x)$ was

$$\left[\frac{\underset{\sim}{P}^2}{2m} + \underset{\sim}{V} \right] \varphi_k = \frac{k^2}{2m} \varphi_k$$

or

$$[\underset{\sim}{P}^2 - k^2]\,|\varphi_k\rangle = -2m\underset{\sim}{V}\,|\varphi_k\rangle$$

where I have written $\varphi_k(x)$ as a ket $|\varphi_k\rangle$. Let the exponential associated with the original plane wave be represented by

$$e^{ikz} = e^{i\vec{k}_0\cdot\vec{x}} \longrightarrow |k_0\rangle$$

The full wavefunction can be written asa sum of this exponential and the scattered wave

$$|\varphi_k\rangle = |k_0\rangle + |\delta\varphi_k\rangle$$

Since

$$[\underset{\sim}{P}^2 - k^2]\,|k_0\rangle = 0$$

We find the following operator equation for the scattered wave:

$$[\underset{\sim}{P}^2 + 2m\underset{\sim}{V} - k^2]\,|\delta\varphi_k\rangle = -2m\underset{\sim}{V}\,|k_0\rangle$$

It is very tempting to write the solution of this equation as solve this equation using the matrix inverse of the operator on the left-hand side. Thus,

$$|\delta\varphi_k\rangle = [\underset{\sim}{P}^2 - k^2 + 2m\underset{\sim}{V}]^{-1}(-2m\underset{\sim}{V})\,|k_0\rangle$$

This solution can even be made completely explicit. To do this, we only have to note that the matrix inverse of an operator is the corresponding Green's function. That is, the equation for a Green's function

$$\mathcal{O}_x \, G(x,y) = \delta^{(3)}(\vec{x}-\vec{y})$$

is just the matrix element between $\langle x|$ and $|y\rangle$ of the matrix equation

$$\underset{\sim}{\mathcal{O}} \cdot \underset{\sim}{G} = \underset{\sim}{1}$$

with

$$G(x,y) = \langle x| \, \underset{\sim}{G} \, |y\rangle$$

The right-hand side is given by the identity

$$\langle x| \, \underset{\sim}{1} \, |y\rangle = \delta^{(3)}(\vec{x}-\vec{y})$$

We can now rewrite the Green's function of the full Schrödinger operator as an expansion in the potential. To do this, use the matrix identity

$$[A+B]^{-1} = A^{-1} - A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} - \cdots$$

This identity looks like the usual expansion of $1/(1+x)$, but there is a subtlety; we must verify that the identity is still true if A and B do not commute. Fortunately, this is not hard to check.

$$(A+B)\,[A+B]^{-1} = 1 - BA^{-1} + BA^{-1}BA^{-1} - \ldots$$
$$\qquad + BA^{-1} - BA^{-1}BA^{-1} + \cdots$$
$$= 1 \quad \checkmark$$

Applying this identity to our solution for $|\delta\varphi_k\rangle$, we find

$$|\delta\varphi_k\rangle = [\underset{\sim}{P}^2 - k^2]^{-1}(-2m\underset{\sim}{V})|k_0\rangle$$
$$- [\underset{\sim}{P}^2 - k^2]^{-1}(-2m\underset{\sim}{V})[\underset{\sim}{P}^2 - k^2]^{-1}(-2m\underset{\sim}{V})|k_0\rangle + \ldots$$

We can make this formula almost totally explicit by inserting complete sets of position states $|y\rangle$, $|z\rangle$, etc.

$$\delta\varphi_k(x) = \int d^3y \; \langle x|[\underset{\sim}{P}^2 - k^2]^{-1}|y\rangle(-2mV(\vec{y}))\, e^{i\vec{k}_0\cdot\vec{y}}$$
$$- \int d^3y\, d^3z \; \langle x|[\underset{\sim}{P}^2 - k^2]^{-1}|y\rangle(-2mV(y))\langle y|[\underset{\sim}{P}^2 - k^2]^{-1}|z\rangle(-2mV(z))\, e^{i\vec{k}_0\cdot\vec{z}}$$
$$+ \ldots$$

To complete the evaluation, we note that the matrix element of $[\underset{\sim}{P}^2 - k^2]^{-1}$ is just the Green's function discussed at the beginning of this lecture.

This series for $\delta\varphi_k(x)$ is called the *Born series.* This series directly leads to a series expansion for the scattering amplitude. The first term in these series is called the *first Born approximation.* This is usually the simplest approximation available for a problem in potential scattering.

Before we study this approximation in detail, I should make one more comment on the Green's function of the free-particle Schrödinger equation. The Green's function of the operator $[\underset{\sim}{P}^2 + a^2]$ has just the form of the Yukawa potential,

$$\langle x|[\underset{\sim}{P}^2 + a^2]^{-1}|y\rangle = \frac{1}{4\pi|\vec{x}-\vec{y}|}\, e^{-a|\vec{x}-\vec{y}|}$$

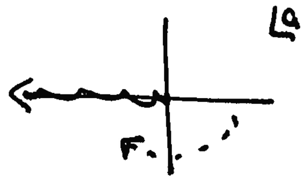
The condition $G(x, y) \to 0$ as $|\vec{x}| \to \infty$ gives this solution uniquely. However, we saw that, when we replace the constant a^2 by $(-k^2)$, we need some additional information to obtain a unique result:

$$[-k^2]^{1/2} = -ik \quad \underline{\text{not}} \quad ik$$

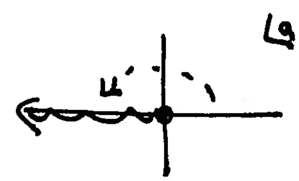
Sometimes, we write

$$G(x,y) = \langle x | \, [\underset{\sim}{P}^2 - k^2 - i\varepsilon]^{-1} \, | y \rangle$$

to indicate that we use the analytic continuation



<u>not</u>



Notice that this corresponds to evaluating the energy of the quantum particle at

$$E = \frac{k^2}{2m} + i\varepsilon$$

which is just *above* the branch cut in the complex E plane, as I recommended in the previous lecture.

I will now return to concrete calculations. In particular, it is time to turn the formal Born series into an explicit expression for the scattering amplitude. In the first Born approximation,

$$\delta\varphi_k(x) = \int d^3y \, \frac{1}{4\pi|\vec{x}-\vec{y}|} e^{ik|\vec{x}-\vec{y}|} \left(-2mV(y)\right) e^{ik\hat{z}\cdot\vec{y}}$$

We need to take the limit of this expression as $|\vec{x}| \to \infty$. In this limit

$$|\vec{x}-\vec{y}| = [x^2 - 2\vec{x}\cdot\vec{y} + y^2]^{\frac{1}{2}}$$

$$= |\vec{x}| \left(1 - \frac{\vec{x}\cdot\vec{y}}{|\vec{x}|^2} + \frac{1}{2}\frac{y^2}{|\vec{x}|^2} - \frac{1}{8}\frac{(2\vec{x}\cdot\vec{y})^2}{|\vec{x}|^4} + \ldots\right.$$

or, writing $r = |\vec{x}|$,

$$|x-y| = r - \hat{r}\cdot\vec{y} + \frac{1}{2}\frac{y^2}{r} - \frac{1}{2}\frac{(\hat{r}\cdot\vec{y})^2}{r} + \ldots$$

As $r \to \infty$, the terms of order y^2 disappear, but the term linear in y remains. Then this leading term is

$$\delta\varphi_k = \frac{e^{ikr}}{r} \cdot \int d^3y \, \frac{1}{4\pi} e^{-ik\hat{r}\cdot\vec{y}} e^{ik\hat{z}\cdot\vec{y}} \left(-2m V(y)\right)$$

The integral gives our approximation to $f_k(\theta, \phi)$,

$$f_k(\theta,\phi) = -\frac{m}{2\pi} \int d^3y \, e^{-i\vec{Q}\cdot\vec{y}} V(y)$$

or

$$f_k = -\frac{m}{2\pi} \tilde{V}(\vec{Q})$$

where $\tilde{V}(\vec{Q})$ is the Fourier transform of the potential and

$$\vec{Q} = k\hat{r} - k\hat{z}$$

is the momentum transferred to the scattered particle.

The final result is very simple and intuitively appealing. The scattering cross section is

$$\frac{d^2\sigma}{d^2\Omega} = \frac{m^2}{(2\pi)^2} \left|\hat{V}(Q)\right|^2$$

The units are m^2 on the left-hand side and $kg^2J^2m^6$ on the right-hand side. Dividing by 4 factors of $\hbar$ makes the units work out correctly. The final result, written in terms of $d\cos\theta$ in a way that can be evaluated numerically, is

$$\frac{d\sigma}{d\cos\theta} = \frac{m^2}{2\pi\hbar^4} \left|\tilde{V}(Q)\right|^2$$

It is instructive to evaluate this Born approximation for the case of a Yukawa potential.

$$V(r) = \frac{g^2}{4\pi r} e^{-ar}$$

We already know the Fourier transform, so we can immediately write

$$\frac{d\sigma}{d\cos\theta} = \frac{g^4 m^2}{2\pi} \frac{1}{(a^2 + |Q|^2)^2}$$

To find the result for the Coulomb potential, take $a \to 0$ and set $g^2 = q_1 q_2/\epsilon_0$. Then

$$\frac{d\sigma}{d\cos\Theta} = \frac{q_1^2 q_2^2 m^2}{2\pi \epsilon_0^2} \frac{1}{(|\vec{Q}|^2)^2}$$

This has the correct units if $\vec{Q}$ is measured in momentum units rather than m^{-1}. We can make this more explicit by writing

$$\begin{aligned} |\vec{Q}|^2 &= |k(\hat{r} - \hat{z})|^2 = k^2(1 - 2\hat{r}\cdot\hat{z} + 1) \\ &= 2k^2(1-\cos\Theta) = 4k^2 \sin^2\Theta/2 \\ &= 8mE\sin^2\Theta/2 \end{aligned}$$

Then, finally, for scattering from a Coulomb potential

$$\frac{d\sigma}{d\cos\Theta} = \frac{q_1^2 q_2^2}{128\pi\epsilon_0^2 E^2} \frac{1}{\sin^4\Theta/2}$$

This is exactly the result that we found in the homework from a purely classical derivation.

The story of the Coulomb potential is even stranger. Not only is the Born approximation in agreement with the classical result for the cross setion, but also it agrees with the exact quantum-mechanical result.

All of this is very lucky. In 1909, Hans Geiger and Ernest Marsden were working in Ernest Rutherford's lab in Manchester, studying the scattering of alpha particles from thin metal foils. Rutherford suggested that they look for backward scattering of alphas and, indeed, they found scattering at large angles. This result was completely at odds with J. J. Thomson's model of the atom and proved, instead, that the atom contained a hard core with large electric charge. Rutherford then used the formula above – now called the *Rutherford formula* – to show that the electric charge of this nucleus was approximately equal to the atomic number. Rutherford had only the classical derivation of the equation, since quantum mechanics had not yet been invented. Fortunately, it is exactly the right answer.

This exactness does not hold for charge distributions that are not point charges. However, for Coulomb scattering, the scattering amplitude is proportional to $1/E$, and, thus, the scattering is weak and the Born approximation is good for scattering of high energy projectiles.

We can use the Born approximation, then, to analyze the scattering of a charged particle from a distribution of charge. The potential is

$$V(\vec{x}) = \int d^3w \quad q_1 \frac{1}{4\pi\epsilon_0 |\vec{x}-\vec{w}|} \rho(\vec{w})$$

where

$$\int d^3w \; \rho(w) = q_2$$

The Fourier transform of $V(\vec{x})$ works out to be

$$\tilde{V}(\vec{Q}) = \int d^3x \, e^{-i\vec{Q}\cdot\vec{x}} \cdot \int d^3w \, \frac{q_1}{4\pi\epsilon_0 |\vec{x}-\vec{w}|} \rho(\vec{w})$$

$$= \int d^3(x-w) \, e^{-i\vec{Q}\cdot(\vec{x}-\vec{w})} \frac{q_1}{4\pi\epsilon_0 |\vec{x}-\vec{w}|} \cdot \int d^3w \; e^{-i\vec{Q}\cdot\vec{w}} \rho(\vec{w}) = \frac{q_1}{\epsilon_0 |\vec{Q}|^2} \tilde{\rho}(\vec{Q})$$

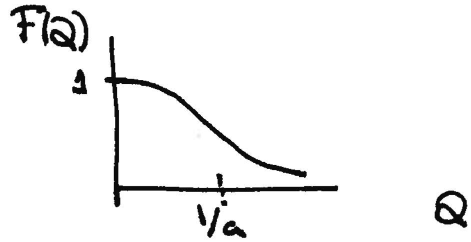
where $\tilde{\rho}(\vec{Q})$ is the Fourier transform of the charge distribution. The Born approximation formula for the cross section is now easy to construct. That formula is usually written

$$\frac{d\sigma}{d\cos\theta} = \frac{q_1^2 q_2^2}{128\pi E^2} \cdot \frac{1}{\sin^4\theta/2} \cdot |F(\vec{Q})|^2$$

where

$$F(\vec{Q}) = \tilde{\rho}(\vec{Q})/q_2$$

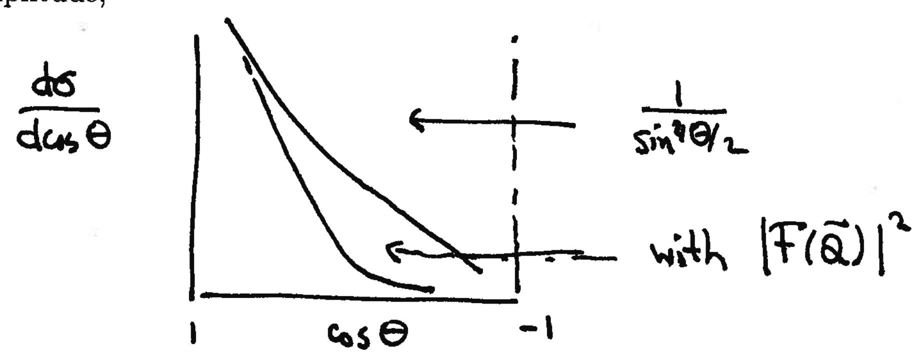
is called the *form factor*. For a localized charge distribution of size a, the form factor falls off with a range $Q \sim \hbar/a$.



The value of the form factor at $Q = 0$ is

$$F(Q=0) = \frac{\int d^3w\, \rho(w)}{q_2} = 1$$

At high values of Q, the form factor $F(Q)$ goes to zero. This suppresses the scattering amplitude,



In principle, we can measure the cross section and extract the form factor by comparing to the result for Coulomb scattering from a point source. Then we can Fourier transform the charge distribution back to real space and determine the shape of the scatterer.

Here at Stanford in the 1950's, Robert Hofstadter used a linear electron accelerator located under what is now the new Engineering Quad to shoot high-energy electrons at nuclei and measure their sizes. The first figure shows a measurement of electron scattering from gold from the letter to the editor of the Physical Review by Hofstadter, Fechter, and McIntyre, Phys. Rev. 91, 422 (1953). The second line of figures showed results from the later paper, Hahn, Ravenhall, and Hofstadter, Phys. Rev. 101, 1131 (1956), and the reconstruction in this paper of the nuclear charge density in a variety of nuclei. The electron energies in these experiments were of the order of 150 MeV, so relativistic formulae were needed, but the principle of the work is the same. Hofstadter won the Nobel Prize for these beautiful experiments.

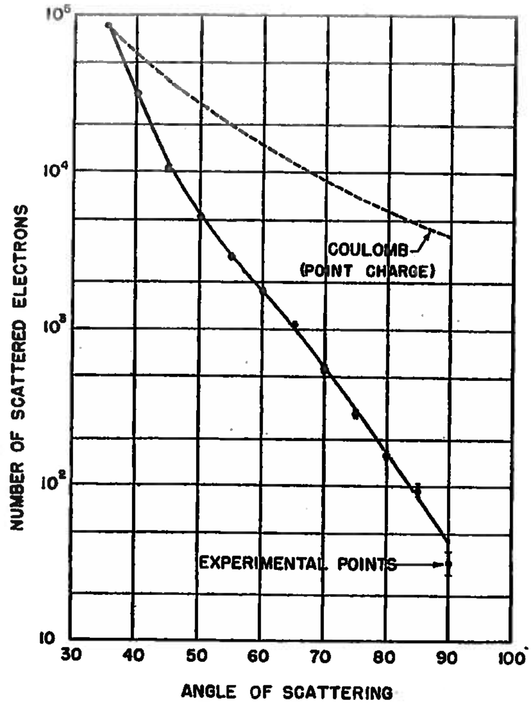


FIG. 3. Typical angular distribution obtained at 116 Mev with a 0.002-inch gold foil. The gold foil was oriented at 45° with respect to the incident beam for all angular settings of the spectrometer magnet.

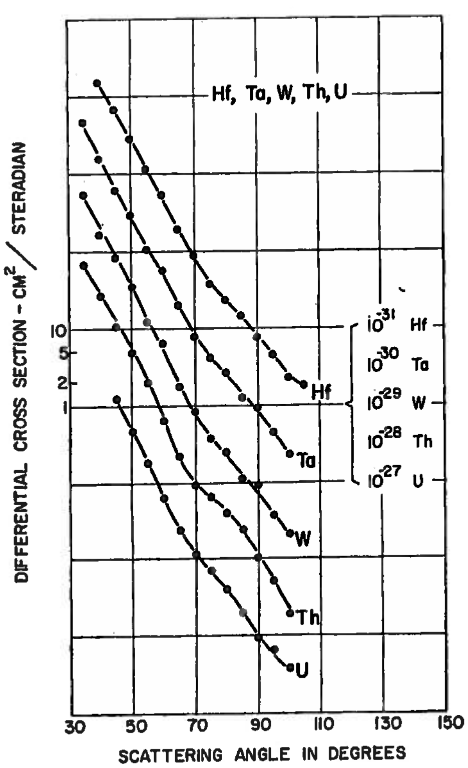


FIG. 13. Experimental cross sections at 183 Mev for the nuclei Hf, Ta, W, Th, and U. Absolute cross sections have been obtained from the counting rate ratio with respect to gold, and from the absolute cross section for gold given in Fig. 3. The dashed lines are smooth curves connecting the experimental points, and are *not theoretical*. The curves have been shifted vertically by factors of ten as indicated.

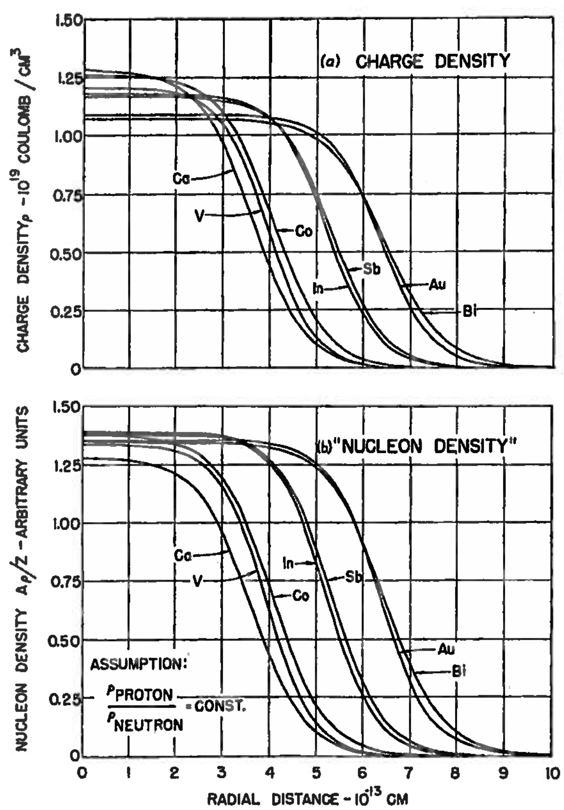


FIG. 14. (a) Charge distributions $\rho(r)$ for Ca, V, Co, In, Sb, Au, and Bi. They are Fermi smoothed uniform shapes, with the parameters given in Table III, and yield the cross sections shown in Figs. 3 and 8–12. (b) A plot of $(A/2Z)\rho(r)$ for the above nuclei. On the assumption that the distribution of matter in the nucleus is the same as the distribution of charge, this represents the "nucleon density."