

May 10

## Interaction of a Schrödinger Particle with Electromagnetic Fields

The first use I would like to make of Fermi's Golden Rule is the computation of the rates for atoms to absorb and emit photons. To prepare for this, however, we need a little more formalism. We need to know how Schrödinger particles couple to electromagnetic fields. For a complete treatment, we should also discuss how to quantize the electromagnetic field and express it in terms of operators that create and destroy photons. In this course, though, I will cheat by working from formulae for classical electromagnetic waves. Still, we must discuss how these waves enter the Schrödinger equation.

If we are dealing only with static electric fields, we can easily write the coupling to a Schrödinger particle. We construct the electrostatic potential  $\Phi$ , such that

$$\vec{E} = -\nabla\Phi$$

Then, for a particle of charge  $q$ , the potential energy is

$$V = q\Phi(\vec{r})$$

and we can write

$$H = \frac{p^2}{2m} + q\Phi$$

What if we have a magnetic field  $\vec{B}$  or a non-static  $\vec{E}$  field? This is obviously more tricky. The Hamiltonian must reproduce the Lorentz force, which depends on the velocity and acts in a direction perpendicular to  $\vec{B}$ . This force cannot be derived from a potential energy; it needs another description.

You might remember the solution from your mechanics course. First, we need to describe the  $\vec{B}$  and  $\vec{E}$  fields by potentials. This is done by adding to  $\Phi$  a vector potential  $\vec{A}$ . The potentials generate the  $\vec{B}$  and  $\vec{E}$  fields by

$$\vec{B} = \nabla \times \vec{A} \quad \vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}$$

Then, I claim, the Hamiltonian that gives the correct particle equations of motion is

$$H = \frac{(\vec{p} - q\vec{A})^2}{2m} + q\Phi$$

where  $\Phi$  and  $\vec{A}$  are evaluated at the position  $\vec{x}(t)$  of the particle. In quantum mechanics,  $\vec{x}(t)$  becomes an operator. This Hamiltonian generates the Heisenberg picture equation of motion for  $\vec{x}(t)$  and  $\vec{p}(t)$

$$i\dot{x}^k = [x^k, H] = i \frac{1}{m} (p^k - qA^k)$$

$$i\dot{p}^k = [p^k, H] = -i \frac{1}{2m} \{ (p^j - qA^j), -q\nabla^k A^j \} - iq\nabla^k \Phi$$

The consequences of these equations are not very obvious. We can write

$$\vec{p} = m\dot{\vec{x}} + q\vec{A}$$

which implies that  $\vec{p}$  is no longer equal to  $m\vec{v}$  but, in addition, contains information about the magnetic field. In any case, we can compute  $\ddot{\vec{x}}(t)$ ,

$$m\ddot{x}^k = \dot{p}^k - q\dot{A}^k$$

$$\begin{aligned}
m\ddot{x}^k &= \frac{q}{2} \{ \dot{x}^j, \nabla^k A^j \} - q \nabla^k \Phi - \frac{d}{dt} A^k(x(t), t) \\
&= \frac{q}{2} \{ \dot{x}^j, \nabla^k A^j \} - q \nabla^k \Phi - \frac{\partial}{\partial t} A^k - \frac{1}{2} \{ \dot{x}^j, \nabla^j A^k \} \\
&= q E^k + \frac{q}{2} \{ \dot{x}^j, (\nabla^k A^j - \nabla^j A^k) \}
\end{aligned}$$

We can recognize

$$\nabla^k A^j - \nabla^j A^k = \epsilon^{kjl} (\nabla \times A)^l = \epsilon^{kjl} B^l$$

so that this equation becomes

$$m\ddot{x}^k = q E^k + \frac{q}{2} \epsilon^{kjl} \{ \dot{x}^j, B^l \}$$

or, ignoring for a moment that  $\vec{x}$  and  $\dot{\vec{x}}$  do not commute,

$$m\ddot{\vec{x}} = q \vec{E} + q \vec{v} \times \vec{B}$$

The Heisenberg equation of motion for  $\vec{x}(t)$  then just gives the Lorentz force law.

Here is a useful exercise that shows that we are on the right track. Consider a Schrödinger particle in a constant magnetic field. This field can be represented by a vector potential

$$\vec{A} = -\frac{1}{2} \vec{r} \times \vec{B}$$

Then

$$\begin{aligned} H &= \frac{1}{2m} (\vec{p} + \frac{q}{2} \vec{r} \times \vec{B})^2 \\ &= \frac{1}{2m} p^2 + \frac{q}{4m} (\vec{p} \cdot \vec{r} \times \vec{B} + \vec{r} \times \vec{B} \cdot \vec{p}) + \frac{q^2}{8m} (\vec{r} \times \vec{B})^2 \end{aligned}$$

In this expression,

$$\vec{p} \cdot \vec{r} \times \vec{B} = \vec{r} \times \vec{B} \cdot \vec{p} = -\vec{r} \times \vec{p} \cdot \vec{B} = -\vec{B} \cdot \vec{L}$$

so that, finally

$$H = \frac{p^2}{2m} + \left(\frac{-q}{2m}\right) \vec{B} \cdot \vec{L} + \frac{q^2}{8m} B^2 r_{\perp}^2$$

The second term here is an interaction of  $\vec{B}$  with the magnetic dipole moment that is set up by an orbiting charge

$$\vec{\mu} = \frac{q\vec{L}}{2m}$$

This effect is called *paramagnetism*. An atom with internal orbital angular momentum and therefore a magnetic moment can lower its energy in a magnetic field by orienting its moment parallel to the applied field. The third term is the dominant effect if an atom has no magnetic moment. This effect is called *diamagnetism*; it is an increase in energy for a nonrotating system of charged particles in a magnetic field.

If we express  $\vec{L}$  as a *dimensionless* quantity, as I will do from here on, the magnetic moment is written

$$\vec{\mu} = \frac{q\hbar}{2m} \vec{L}$$

The magnitude of the moment depends on the charge and mass of the particle. For an electron, the quantum of magnetic moment is the *Bohr magneton*,

$$\frac{e\hbar}{2m_e} = 5.788 \times 10^{-5} \text{ eV/Tesla}$$

In addition to their orbital angular momentum, particles have an intrinsic spin, and this also contributes to the magnetic moment. The relation is written

$$\vec{\mu}_s = g \frac{q\hbar}{2m} \vec{S}$$

where  $g$  is called the *Landé g-factor*. This factor is a property of the particle in question,

$$g = \begin{cases} 2.0032 & \text{for the electron w. } g = -e \\ 5.5856 & \text{for the proton w. } g = e \\ -3.8261 & \text{for the neutron, w. } g = e \end{cases}$$

For the electron, we have a fundamental understanding of this value. The relativistic equation for a spin  $\frac{1}{2}$  particle, the Dirac equation, predicts  $g = 2$ . Quantum corrections to the structure of the electron give a small correction to this value,

$$g = 2 \left( 1 + \frac{\alpha}{\pi} + \dots \right)$$

For the proton and neutron, the large deviations from the expected values  $g = 2$  and  $g = 0$  tells us that these are not elementary particles. In fact, it is known that they are composite, being bound states of quarks. The quark model, in which the proton and neutron are bound states of three quarks, gives a reasonable semi-quantitative understanding of the values of the proton and neutron  $g$ -factors.

The full Hamiltonian coupling a Schrödinger particle to electric and magnetic fields can then be written

$$H = \frac{(\vec{p} - q\vec{A})^2}{2m} + q\Phi - \vec{B} \cdot \vec{\mu}_s$$

or, for a constant  $\vec{B}$  field,

$$H = \frac{\vec{p}^2}{2m} + q \frac{\vec{B}^2 r_{\perp}^2}{8m} - \vec{B} \cdot \frac{q}{2m} (\vec{L} + g\vec{S})$$

It is odd that the Schrödinger equation involves  $\vec{A}$  and not  $\vec{B}$ . The principle that drives this is called the *principle of local gauge invariance*, and it merits some discussion. Maxwell's equations are written in terms of  $\vec{E}$  and  $\vec{B}$ . This means that they do not involve all of the possible degrees of freedom in  $\Phi$  and  $\vec{A}$ . We can make this explicit by noting that if we transform  $\Phi$  and  $\vec{A}$  by

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\alpha \quad \Phi \rightarrow \Phi - \frac{\partial\alpha}{\partial t}$$

this transformation leaves  $\vec{E}$  and  $\vec{B}$  invariant for any function  $\alpha(\vec{x}, t)$  with arbitrary variation over space and time. This transformation of the potentials is called a *local gauge transformation*. If we write Maxwell's equations in terms of  $\Phi$  and  $\vec{A}$ , we can see directly that they are invariant to local gauge transformations.

Since the Schrödinger equation involves  $\Phi$  and  $\vec{A}$ , it is not obviously locally gauge invariant. Even if only  $\Phi$  is nonzero, the Schrödinger equation changes when we make a gauge transformation with a time-dependent  $\alpha$ .

In discussing equations of motion, we do not need to demand that these equations are *invariant* under a symmetry. Rather, what we want is that the equation are

*covariant*, that is, if a set of fields satisfy the equations, the symmetry-transformed fields must also, automatically, be solutions of the equations. In this case, we say that the equations are *covariant* with respect to the symmetry. I claim that the Schrödinger equation has this property with respect to local gauge transformations. If we supplement the above transformations of  $\Phi$  and  $\vec{A}$  with the transformation of  $\psi$

$$\psi(x,t) \rightarrow e^{iq\alpha(x,t)} \psi(x,t)$$

then, I claim, if the original fields  $(\psi, \Phi, \vec{A})$  satisfy the Schrödinger equation, the transformed field must also satisfy this equation.

A nice way to prove this is to define the generalized derivatives

$$D^0 = \left( \frac{\partial}{\partial t} + iq\Phi \right) \quad D^k = \left( \frac{\partial}{\partial x^k} - iqA^k \right)$$

These objects are called *covariant derivatives*. The Schrödinger equation with electromagnetic potentials  $\Phi$  and  $\vec{A}$

$$i \frac{\partial}{\partial t} \psi = \left[ \frac{1}{2m} (-i\nabla^k - qA^k)^2 + q\Phi \right] \psi$$

can be rewritten using the covariant derivatives,

$$i D^0 \psi = - \frac{1}{2m} (D^k)^2 \psi$$

The transformation of  $D^0\psi$  under a local gauge transformation is

$$\begin{aligned}
D^0 \psi &\rightarrow \left( \frac{\partial}{\partial t} + i q \Phi - i q \frac{\partial \alpha}{\partial t} \right) e^{i q \alpha} \psi \\
&= e^{i q \alpha} \left( \frac{\partial}{\partial t} + i q \Phi - i q \frac{\partial \alpha}{\partial t} + i q \frac{\partial \alpha}{\partial t} \right) \psi \\
&= e^{i q \alpha} \left( \frac{\partial}{\partial t} + i q \Phi \right) \psi = e^{i q \alpha} D^0 \psi
\end{aligned}$$

Similarly,

$$D^k \psi \rightarrow e^{i q \alpha} (D^k \psi)$$

In general, the property of a covariant derivative is that, if  $f(x)$  has a given transformation property under a symmetry, the covariant derivative  $Df(x)$  has the same transformation property.

Using this property of the covariant derivatives, we see that the Schrödinger equation written above transforms under a local gauge transformation to

$$e^{i q \alpha(x,t)} \cdot i D^0 \psi - e^{i q \alpha(x,t)} \left[ -\frac{1}{2m} (D^k)^2 \psi \right] = 0$$

Then, if the original  $(\psi, \Phi, \vec{V})$  satisfied the Schrödinger equation, the transformed fields also must satisfy the Schrödinger equation. The requirement of local gauge invariance dictates the placement of  $\Phi$  and  $\vec{A}$  in the Schrödinger equation.

This argument sounds more profound if it is run backwards. The phase of the Schrödinger wavefunction is not measurable, so it is interesting to consider the possibility that variation of the Schrödinger wavefunction by a space-time dependent phase might be a fundamental symmetry of Nature. Postulate, then, that the transformation

$$\psi(x,t) \rightarrow e^{i \beta(x,t)} \psi(x,t)$$

is a symmetry. This symmetry makes it difficult to write a field equation for  $\psi(\vec{x}, t)$ , because a term with  $\partial/\partial t$  or  $\partial/\partial x^k$  would not necessarily be covariant. To make these terms covariant, we need to replace the ordinary derivatives by covariant derivatives, defined above, with the fields  $\Phi, \vec{A}$  assigned the transformation laws

$$\vec{A} \rightarrow \vec{A} + \frac{1}{q} \vec{\nabla} \beta \quad \Phi \rightarrow \Phi - \frac{1}{q} \frac{\partial}{\partial t} \beta$$

Notice that  $(\Phi, \vec{A})$  form a 4-vector, and that the transformation has a simple form when written in relativistic notation

$$A_\mu \rightarrow A_\mu - \frac{1}{q} \partial_\mu \beta \quad \text{for } A^\mu = (\Phi, \vec{A})$$

Finally, we need the equations of motion for the newly introduced fields  $\Phi$  and  $\vec{A}$ . The unique set of linear relativistically invariant field equations that is covariant under local gauge transformations is Maxwell's equations. In relativistic notation, the  $\vec{E}$  and  $\vec{B}$  fields form the tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

and Maxwell's equations take the form

$$0 = \partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu$$

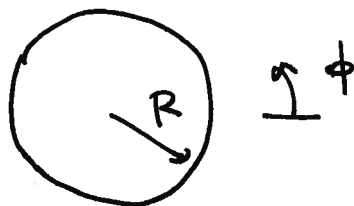
Thus, we can explain Maxwell's equations by arguing that they follow from the principle of local gauge invariance.

Local gauge invariance can be extended to more general transformations, not only changes of phase but also transformations through larger unitary groups. Applying the local just given to such extended transformations leads to the correct equations

of motion for the fields that give rise to the weak and strong interactions of particle physics. Local gauge invariance is also closely related to the principle of general covariance, the governing principle of Einstein's theory of gravity.

There is a strange consequence of coupling of the Schrödinger wavefunction to  $\vec{A}$  instead of  $\vec{B}$ . To introduce this, I will first discuss the problem of a Schrödinger particle confined to a circular wire in a constant magnetic field. This system is actually realizable at macroscopic size using a superconducting wire. The effect of the magnetic field that I will describe gives the principle of operation of the SQUID magnetometer.

Consider first a circular wire of radius  $R$  in zero magnetic field



The Hamiltonian for a particle on the wire is

$$H = -\frac{1}{2m} \frac{1}{R^2} \frac{\partial^2}{\partial \phi^2} \psi$$

The eigenstates of  $H$  satisfying periodic boundary conditions are

$$\psi = e^{in\phi}$$

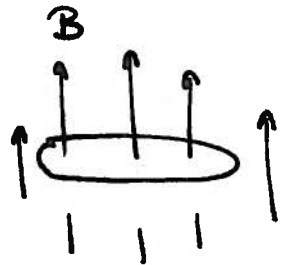
with energies

$$E = \frac{n^2}{2mR^2}$$

These states have orbital angular momentum

$$L^2 = n$$

Now turn on a constant magnetic field perpendicular to the plane of the wire. Using the Hamiltonian written above, the energies of the states are shifted to

$$E = \frac{n^2}{2mR^2} - \frac{q}{2m} \vec{B} \cdot \vec{L} + \frac{q^2 B^2 R^2}{8m}$$


This is a perfect square,

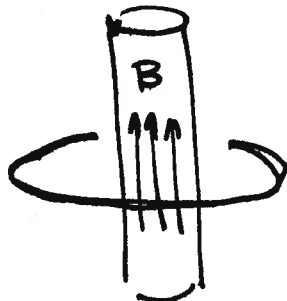
$$E = \frac{1}{2mR^2} \left( n^2 - qnBR^2 + \frac{q^2 B^2 R^4}{4} \right) = \frac{1}{2mR^2} \left( n - \frac{qBR^2}{2} \right)^2$$

The spectrum of states shifts in a systematic way. If

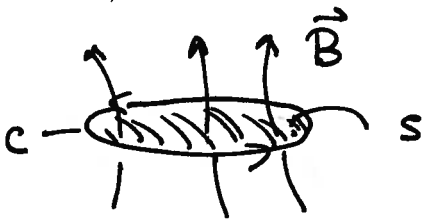
$$\frac{q}{2m} BR^2 = 1$$

the states move over by one step, so that the minimum energy is now  $E = 0$  at  $n = 1$ .

All of this makes good physical sense. However, it is interesting to consider a similar problem in which the magnetic field does not actually touch the Schrödinger particle. Instead of turning on a constant magnetic field, thread the circle of wire with a long solenoid



with  $\vec{B}$  nonzero inside the solenoid but  $\vec{B} = 0$  outside. There is still an  $\vec{A}$  field outside the solenoid. By Green's theorem, if  $\vec{A} = \vec{\nabla} \times \vec{B}$ , then

$$\int_S d\vec{a} \hat{n} \cdot \vec{B} = \oint_C d\vec{x} \cdot \vec{A}$$


The diagram shows a cylindrical solenoid with magnetic field lines  $\vec{B}$  pointing upwards. A surface  $S$  is drawn as a shaded oval around the solenoid, with its boundary  $C$  indicated by a dashed line. The surface normal  $\hat{n}$  is shown pointing outwards from the surface.

where  $S$  is a surface in space and  $C$  is the curve that is the boundary of the surface. So, if there is a nonzero flux through the solenoid

$$\Phi_0 = \int_S d\vec{a} \hat{n} \cdot \vec{B}$$

then there is a nonzero  $\vec{A}$  outside the solenoid. For a cylindrically symmetric situation

$$\vec{A} = \hat{\phi} \frac{\Phi_0}{2\pi r} \quad \hat{\phi} = (-\sin\phi, \cos\phi, 0)$$

It is easy to check that

$$\oint_C d\vec{x} \cdot \vec{A} = \Phi_0$$

and one can also see that

$$\vec{\nabla} \times \vec{A} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times \left( -\frac{y}{2\pi r^2}, \frac{x}{2\pi r^2}, 0 \right) \Phi_0 = 0$$

If we put this  $\vec{A}$  into the Hamiltonian, the Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2m} \left( -i \frac{1}{R} \frac{\partial}{\partial \phi} - q \frac{\Phi_0}{2\pi R} \right)^2$$

The wavefunction

$$e^{in\phi}$$

is an eigenstate, with energy eigenvalue

$$E = \frac{1}{2mR^2} \left( n - q \frac{\Phi_0}{2\pi} \right)^2$$

For

$$\Phi_0 = \pi R^2 \cdot B$$

this agrees perfectly with our calculation for a constant  $B$  field. In the earlier discussion, we interpreted the result as the interaction of the Schrödinger particle with the magnetic field. However, we now see that we obtain the same result if the magnetic field is *zero* where the particle is, as long as the  $\vec{A}$  field at the particle's location correctly reflects the magnetic flux inside the loop. This response of a particle to the  $\vec{A}$  field in a setup with  $\vec{B} = 0$  is called the *Bohm-Aharonov effect*.

You might have been told that the  $\vec{E}$  and  $\vec{B}$  fields describe the complete physical content of the  $\Phi$  and  $\vec{A}$  fields. This is not quite correct. The correct statement is not so simple: Physical quantities can depend on any locally gauge invariant observable built from  $\Phi$  and  $\vec{A}$ . In particular, the quantity

$$\oint d\vec{x} \cdot \vec{A}$$

is invariant under local gauge transformations, since, if  $C$  is a closed contour and  $\alpha$  has a definite value at each  $(\vec{x}, t)$ ,

$$\oint_C d\vec{x} \cdot \vec{A} \rightarrow \oint_C d\vec{x} \cdot \vec{A} + \oint_C d\vec{x} \cdot \vec{\nabla} \alpha = \oint_C d\vec{x} \cdot \vec{A} + 0$$

The Bohm-Aharonov energy shift depends on this quantity.

To discuss the emission and absorption of light, we need to write reference definite formulae for an electromagnetic plane wave. This is a plane wave solution of Maxwell's equations in free space

$$\begin{aligned} \epsilon_0 \vec{\nabla} \cdot \vec{E} &= 0 & \vec{\nabla} \cdot \vec{B} &= 0 \\ \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} &= 0 & \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \end{aligned}$$

The two equations on the right are automatically satisfied when we write  $\vec{E}$  and  $\vec{B}$  in terms of  $\Phi$  and  $\vec{A}$ . To find a plane wave, we need to satisfy the two equations on the left. I claim that this can be done setting  $\Phi = 0$ . A general plane wave form of  $\vec{A}$  is

$$\vec{A}(\vec{x}, t) = \text{Re} \left[ -i A_0 \vec{\epsilon} e^{-i(\omega t - \vec{k} \cdot \vec{x})} \right]$$

From this, we can compute

$$\begin{aligned} \vec{E} &= \text{Re} \left[ A_0 \omega \vec{\epsilon} e^{-i(\omega t - \vec{k} \cdot \vec{x})} \right] \\ \vec{B} &= \text{Re} \left[ A_0 \vec{k} \times \vec{\epsilon} e^{-i(\omega t - \vec{k} \cdot \vec{x})} \right] \end{aligned}$$

The equation

$$\vec{\nabla} \cdot \vec{E} = 0$$

implies

$$\vec{k} \cdot \vec{E} = 0$$

Since  $\epsilon_0 \mu_0 = 1/c^2$ , the equation

$$\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} = 0$$

implies

$$\epsilon_0 c^2 i \vec{k} \times (\vec{k} \times \vec{E}) + i \epsilon_0 \omega^2 \vec{E} = 0$$

Using  $\vec{k} \cdot \vec{E} = 0$ , this reduces to

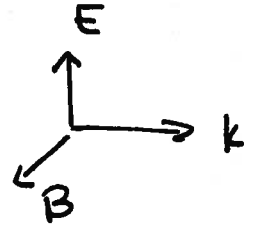
$$-c^2 k^2 + \omega^2 = 0$$

So the expression for  $\vec{A}$  above is a plane wave solution under the conditions

$$\vec{k} \cdot \vec{E} = 0 \quad \omega = kc$$

There are two possible choices for the polarization vector  $\vec{\epsilon}$ . For  $\hat{k}$  in the  $\hat{z}$  direction, we can take these vectors to be

$$\vec{\epsilon} = \hat{1}, \hat{2}$$



However, it is often convenient to use the complex-valued vectors

$$\vec{\epsilon}_+ = \frac{1}{\sqrt{2}} (\hat{1} + i\hat{2}) \quad \vec{\epsilon}_- = \frac{1}{\sqrt{2}} (\hat{1} - i\hat{2})$$

corresponding to right- and left-handed circular polarization. These choices are very useful in quantum mechanics, since they correspond to quanta with angular momentum  $+1$  and  $-1$ , respectively, about the  $\hat{z}$  axis. In all cases, I will impose the condition

$$\vec{\epsilon}^* \cdot \vec{\epsilon} = 1$$

and let the quantity  $A_0$  parametrize the strength of the fields.

The electromagnetic energy density is given by

$$\mathcal{E} = \frac{1}{2} (\epsilon_0 E^2 + \frac{1}{\mu_0} B^2)$$

The flux of electromagnetic energy is given by the Poynting vector

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

If

$$A_0 = |A_0| e^{-i\phi}$$

the energy density of a plane wave evaluates to

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} \epsilon_0 \omega^2 |A_0|^2 |\vec{E}|^2 \cos^2(\omega t - \vec{k} \cdot \vec{x} + \phi) \\ &\quad + \frac{1}{2} \epsilon_0 c^2 k^2 |A_0|^2 |\vec{E}|^2 \cos^2(\omega t - \vec{k} \cdot \vec{x} + \phi) \end{aligned}$$

If we average over a cycle of the oscillation, using

$$\langle \cos^2(\omega t - \vec{k} \cdot \vec{x} + \phi) \rangle = \frac{1}{2}$$

we find

$$\langle \mathcal{E} \rangle = \frac{1}{2} \epsilon_0 \omega^2 |A_0|^2$$

Similarly, the Poynting vector evaluates to

$$\vec{S} = \epsilon_0 c^2 \omega k |A_0|^2 \vec{E} \times (\hat{k} \times \vec{E}) \cos^2(\omega t - kx + \phi)$$

$$\langle \vec{S} \rangle = \frac{1}{2} \epsilon_0 \omega^2 |A_0|^2 \cdot c \cdot \hat{k}$$

To go to quantum mechanics, we will need to divide these electromagnetic waves into photons, elementary quanta with energy

$$E = \hbar \omega$$

We can do this simply by dividing the energy per photon into the energy density and flux above. The density of photons in the electromagnetic plane wave is obtained by dividing into  $\mathcal{E}$

$$\frac{\text{photons}}{m^3} = \frac{\epsilon_0 \omega |A_0|^2}{2}$$

The flux of photons is obtained by dividing into  $\vec{S}$

$$\Phi = \frac{\text{photons}}{m^2 \text{ sec}} = \frac{\epsilon_0 \omega |A_0|^2}{2} \cdot c$$

These results are obviously consistent, since the velocity of a photon is  $c$ .

We are now ready to compute the rates for photon emission and absorption by atoms. I will take up that computation in the next lecture.