Physics 130– Problem Set # 8

(due Wednesday, March 13)

1. The problem concerns the diagonalization of the spin Hamiltonian with nearest neighbor interactions

$$H_N = -J \sum_{j=1}^{N} \vec{S}_j \cdot \vec{S}_{j+1}$$

where H acts on a line of N spin $\frac{1}{2}$ spins and $\vec{S}_j = \hbar\vec{\sigma}_j/2$ acting on the jth spin. Assume periodic boundary conditions, that is, $\vec{S}_{N+1} = \vec{S}_1$. The problem for $N \to \infty$ is called the *one-dimensional Heisenberg ferromagnet*. Let

$$S^a = \sum_{j=1}^{N} S_j^a \qquad a = x, y, z$$

(a) Show that

$$\vec{\sigma}_j \cdot \vec{\sigma}_k = 2[\sigma_j^+\sigma_k^- + \sigma_j^-\sigma_k^+] + \sigma_j^z\sigma_k^z$$

where

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Using this representation, show that $S^z = \sum_j S_j^z$ commutes with H_N for any N.

(b) The case $N = 2$ is easy. The Hilbert space is the 4-dimensional space of 2 spins. Show that

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}[S^2 - S_1^2 - S_2^2]$$

Write the eigenfunctions and eigenvalues of H_2.

(c) Show that the trick used in part (b) also works for the case of 3 spins. In this case, there is an 8-dimensional Hilbert space. We need to diagonalize S^2, but the S^2 eigenstates are the state of definite total spin found in Problem 3 of the previous problem set. Write the eigenfunctions and eigenvalues.

(d) The case of 4 spins needs a more careful analysis. There are a total of 16 states. Sort these according to their total S^z eigenvalues. Show that there are 1, 4, 6, 4, 1 states for $S^z/\hbar = -2, -1, 0, 1, 2$, and enumerate these states. Show explicitly that the states with $S^z = \pm 2$ are eigenstates of H, and compute the corresponding eigenvalues.

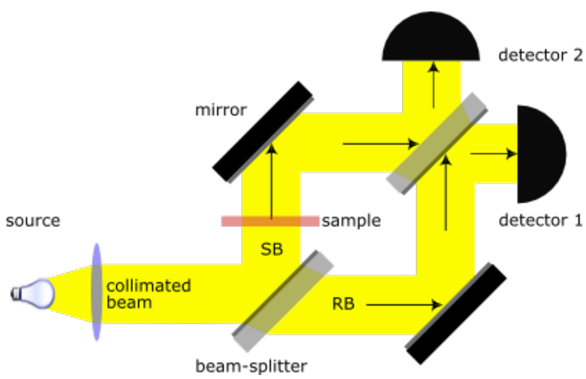
(e) The operation T of translating by one site is a symmetry of this problem: $[T, H_4] = 0$. Since $T^4 = 1$ (why?), the eigenvalues of T are $1, i, -1, -i$. For the states with $S^z = -\hbar$, construct four linear combinations that are eigenstates of T, one with each of these eigenvalues. Show explicitly that these are eigenstates of H_4, and compute the corresponding eigenvalues of H_4. It is interesting to think of the spin flipped up as a particle; it is called a *spin wave*. If we write $T = \exp[iPx_j/\hbar]$, where $x_j = j$ is the position of the jth spin, then P is the momentum of the particle.

(f) Solve for the eigenvectors and eigenvalues for $S^z = +\hbar$ in the same way. The spin flipped down is a *spin hole*.

(g) Finally, we can discuss the problem of the 6 states with $S^z = 0$. Sort these out into linear combinations with definite T. Act H_4 on those states. You will see that T is not changed, and that the hardest problem that you need to solve is the diagonalization of a 2×2. Find the eigenvectors and eigenvalues of H_4 in this sector.

If you lack confidence, or if you would like a nice numerical exercise, type the 16×16 or the 6×6 matrix into Mathematica or MatLab, ask for the eigenvalues, and see if you get the same answer as from the above.

2. Consider the following device, called a *Mach-Zehnder interferometer* [figure courtesy of Wikipedia]



A light wave is injected into the device as indicated. The devices in gray are half-silvered mirrors. The first mirror splits the beam into two equal and coherent parts. These are reflected from mirrors at the corners and recombine at the fourth corner, where there is another half-silvered mirror. If both beams are present and coherent at this mirror, they interfere to produce a beam that travels horizontally to detector 1. No light goes to detector 2. If a sample is introduced as indicated in the figure, this changes the phase relation between the beams. Then light can go to detector 2, with the sum of the intensities going to 1 and 2 equal to the original intensity. If the light intensity is decreased so that only one photon at a time is in the interferometer, it works the same way. When there is no sample, the photon always exits horizontally and is detected by detector 1. When the sample gives the photon's wave form a phase shift of π, the interference is destructive for the horizontal path but constructive for the vertical path, so the photon always exits vertically and is detected by the detector 2.

(a) If the sample gives the photon a phase shift of $\pi/2$, the photon is always exits and is seen by one of the detectors. With what probabilities?

(b) If the sample is replaced by a barrier that stops the photon, what are the possible outcomes, and with what probabilities do they occur?

(c) If the sample is replaced by a bomb that explodes when it is hit by a photon, what are the possible outcomes, and with what probabilities do they occur?

(d) We might have some bombs that are duds and do not interact with the photon. (Assume that they do not even produce a phase shift.) In this case, what are the possible outcomes, and with what probabilities do they occur?

(e) Notice that we can identify some bombs as being live without needing to explode them. What fraction of the live bombs can be identified?

(f) Repeating the procedure with another photon in the ambiguous cases, and continuing until all live bombs are either identified or exploded, what fraction of the live bombs remain?

[Yes, this example was invented by two Israeli physicists, Avshalom Elitzur and Lev Vaidman. Please solve the problem first, then do the web search.]

3. Consider a problem in which two atoms scatter electrons that are fired at them. The atoms are located at $\vec{x}_1 = (a/2, 0, 0)$ and $\vec{x}_2 = (-a/2, 0, 0)$. In the idealization in which the initial waveform of the electron is

$$\psi = e^{ipz/\hbar}$$

the scattered wave from the atom j evaluated at $\vec{x}$ is

$$\psi = \frac{b}{r} e^{ip' r_j/\hbar}$$

where $p' = |\vec{p}'|$ is the momentum of the electron after scattering and $r_j = |\vec{x} - \vec{x}_j|$. Let $r = |\vec{x}|$. If we measure the wave at a distance $r >> a$, it is a good approximation to replace $r_j \approx r - \hat{r} \cdot \vec{x}_j$.

(a) To begin, go back to a simpler case. Consider scattering from one atom at $\vec{x} = 0$. The initial waveform is probably not exactly a plane wave but, rather, a wave packet with large spatial extent L in the $\hat{z}$ direction. Inside the wave packet, the waveform will be that given above for the initial wave, times a normalization constant. The scattered wave is an expanding spherical shell of thickness approximately L, where the waveform inside the shell is that given above for the scattered wave. Compute the total probability in the shell, times the same normalization. Show that the factor $1/r$ in the waveform is needed for this probability to be independent of time.

(b) Now return to the problem of scattering from two atoms. Compute the ratio of the probability in this case for scattering at the angle θ in the $\hat{x}$, $\hat{z}$ plane as a ratio to the probability computed in part (a), assuming that the scattering from the two centers is coherent. (For example, show that, for small θ, the scattering in this case is 4 times larger.) The unscattered beam is at $\theta = 0$, so you should stay away from this point and always assume $\theta > 0$.

(c) The scattering will be coherent if the scattering is *elastic*, that is, if the state of the atoms does not change. Another possibility is that during the scattering of the electron from atom 1, the atom is disrupted in some way, and similarly for the scattering of the electron from the atom 2. In general, in this case, the electron will lose energy to the atom. The scattering is then *inelastic*. If the two states are distinct and distinguishable, compute the probability of scattering at the angle θ as a ratio to the answer in (a). Why is the answer different from that in case (b)?

(d) A third possibility is that the scattering is *inelastic* but *coherent*. For example, the two atoms might be a molecule and the excitation in the scattering process might be to a bonding orbital

$$\frac{1}{\sqrt{2}}[|1\rangle + |2\rangle]$$

where $|1\rangle$ and $|2\rangle$ are identical excited states of the atoms 1 and 2. Compute the scattering probability as a function of θ in this case as a ratio to the answer in part (a).

(e) Alternatively, the excitation in the scattering process might be to an anti-bonding orbital

$$\frac{1}{\sqrt{2}}[|1\rangle - |2\rangle]$$

where $|1\rangle$ and $|2\rangle$ are identical excited states of the atoms 1 and 2. Compute the scattering probability as a function of θ in this case as a ratio to the answer in part (a).