

# Physics 130– Problem Set # 7

(due Wednesday, March 6)

1. This problem works out properties of the vector representation of the rotation group, given by the matrices

$$S^x = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad S^y = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad S^z = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- (a) Find the eigenvectors and eigenvalues of  $S^x$ ,  $S^y$ , and  $S^z$ . Construct  $S^2 = (S^x)^2 + (S^y)^2 + (S^z)^2$ , and compute its eigenvectors and eigenvalues. (Actually, all eigenvalues of  $S^2$  are equal. Why?)
  - (b) For each eigenvector of  $S^z$ , compute the effect of a rotation by  $\phi$  about the  $\hat{z}$  axis.
  - (c) For each eigenvector of  $S^z$ , compute the effect of a rotation by  $\pi/2$  about the  $\hat{y}$  axis. Expand out the result as a linear combination of  $S^z$  eigenvectors.
  - (d) Rotation by  $\pi/2$  about  $\hat{y}$  sends  $z \rightarrow x$ ,  $x \rightarrow -z$ . The inverse of this rotation sends  $z \rightarrow -x$ ,  $x \rightarrow z$ . Write the spherical harmonics  $Y_{1m}(\theta, \phi)$  as functions of  $x$ ,  $y$ , and  $z$ , plug in the latter transformation, and expand the result as a linear combination of spherical harmonics. Compare their transformations to those in part (c). They are the same transformations! Why? [Qualification: You might have to readjust the overall phases that you chose for the eigenvectors of  $S^z$  to make this correspondence precise.]
2. The electron has a magnetic moment, proportional to its spin

$$\vec{\mu} = -g \frac{e}{2m} \vec{S}$$

where  $g = 2$  to 0.1% accuracy and  $\vec{S} = \hbar \vec{\sigma}/2$ . The electron then interacts with a magnetic field according to the Hamiltonian

$$H = -\vec{\mu} \cdot \vec{B}$$

- (a) For  $\vec{B} = B\hat{z}$ , a constant field in the  $\hat{z}$  direction, find the eigenvectors and eigenvalues of  $H$ .
- (b) For  $\vec{B} = B\hat{z}$ , solve the Schrödinger equation for the electron spinor assuming the following initial conditions
  - i.  $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  (spin up along the  $\hat{z}$  axis).

ii.  $\xi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  (spin up along the  $\hat{x}$  axis).

iii.  $\xi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$  (spin up along the  $\hat{y}$  axis).

- (c) How do we interpret the results of part (b)? The spinor state in which the spin points along the axis  $\hat{n}$  can be constructed as the eigenstate with eigenvalue  $+\hbar/2$  of the operator  $\hat{n} \cdot \hbar \vec{\sigma}/2$ . Construct this state for (i)  $\hat{n} = \hat{z}$ , (ii)  $\hat{n} = \cos \theta \hat{z} + \sin \theta \hat{x}$ , (iii) a general unit vector  $\hat{n} = \cos \theta \hat{z} + \sin \theta (\cos \phi \hat{x} + \sin \phi \hat{y})$ .
- (d) With this reference, show that the result of part (b) is that the spin precesses about the direction of the magnetic field. With what frequency?
- (e) Assume we have the ability to turn on a magnetic field of strength  $B$  in the  $\hat{z}$  or in the  $\hat{x}$  direction for some period of time. Define a protocol for starting in the state of spin up along the  $\hat{z}$  axis and ending up in the state that is spin up along the  $\hat{y}$  axis. Define a protocol for starting in the state of spin up along the  $\hat{z}$  axis and ending up in the state that is spin up along the  $\hat{x}$  axis.
3. Once we know how to transform a state with one electron under rotations, we can work out how to transform a state with several electrons under rotations.
- (a) Begin with the case of two electrons. If we ignore the spatial wavefunctions and look only at the spins, the state of two electrons can be described by two spinors. For example

$$|\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

A rotation transforms this state according to

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \mathcal{U}(\vec{\alpha}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathcal{U}(\vec{\alpha}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Expand this expression in powers of  $\vec{\alpha}$  and study the term that is linear in  $\vec{\alpha}$ . This expression gives the infinitesimal generator of the transformation. Show, in this way, that the infinitesimal version of the transformation is

$$|\xi_1 \xi_2\rangle \rightarrow (1 - i\vec{e} \cdot \frac{\vec{\sigma}_1}{2} - i\vec{e} \cdot \frac{\vec{\sigma}_2}{2}) |\xi_1 \xi_2\rangle$$

where  $\sigma_1$  acts on the first spinor and  $\sigma_2$  acts on the second spinor. We can write the generator of this transformation as

$$\vec{S} = \vec{S}_1 + \vec{S}_2 = \hbar \frac{\vec{\sigma}_1}{2} + \hbar \frac{\vec{\sigma}_2}{2}$$

- (b) For a system of 2 electrons, show that the states

$$|\uparrow\uparrow\rangle \quad |\uparrow\downarrow\rangle \quad |\downarrow\uparrow\rangle \quad |\downarrow\downarrow\rangle$$

are eigenstates of  $S^z$  with eigenvalues, respectively,  $\hbar, 0, 0, -\hbar$ .

(c) Show that the state

$$|0\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$$

satisfies  $S^i |0\rangle = 0$  for  $i = x, y, z$ . This is a state of total spin 0.

(d) Then the states

$$|1, 1\rangle = |\uparrow\uparrow\rangle \quad |1, 0\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle] \quad |1, -1\rangle = |\downarrow\downarrow\rangle$$

are plausibly the three states of a spin 1 system. Show that these states are eigenvectors of  $S^2$  with eigenvalue  $S^2 = 2\hbar^2$ .

(e) Notice that the state  $|0\rangle$  is antisymmetric with respect to interchange of the two spins, while the three states  $|1m\rangle$  are symmetric with respect to interchange of the two spins. Show that the interchange operation commutes with  $\vec{S}$ . Then states with different transformation properties under interchange must belong to different irreducible representations.

(f) Next, consider the case of 3 electrons. There are a total of 8 states. Write the 4 states that are totally symmetric under interchange of any two spins. Show that, if

$$\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$$

these 4 states have  $S^z = -\frac{3}{2}\hbar, -\frac{1}{2}\hbar, \frac{1}{2}\hbar, \frac{3}{2}\hbar$ . In fact, they form a spin  $\frac{3}{2}$  representation.

(g) By counting, the remaining 4 states must form two spin  $\frac{1}{2}$  representations. (Why?) In fact, if we combine any two spins to a spin 0 as is done in part (c), the resulting state must be a spin  $\frac{1}{2}$ . Write out the 3 possible states with  $S^z = \frac{1}{2}\hbar$  generated by this prescription. Show explicitly that (i) these states are orthogonal to the four states constructed in (f), and (ii) these 3 states are not linearly independent. Write two orthogonal linear combinations. Each of these is the  $S^z = \frac{1}{2}\hbar$  state of a spin  $\frac{1}{2}$  representation. Write the corresponding  $S^z = -\frac{1}{2}\hbar$  states. These 4 states, plus the 4 in part (f), provide a basis for the 8-dimensional vector space of 3 electrons.