Physics 130– Problem Set # 4

(due Wednesday, Feb. 13)

1. Find the eigenvectors and eigenvalues of the following matrices M. For cases (a) and (b), construct the unitary matrix U such that $U^\dagger M U$ is diagonal, and verify this explicitly. You can diagonalize all of these matrices easily, by typing them in and pushing a button, using Mathematica or Matlab—but please don't. The purpose of this problem is that you should learn how to solve these problems by hand.

   (a) $M = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$

   (b) $M = \begin{pmatrix} 3 & 2i \\ -2i & 0 \end{pmatrix}$

   (c) $M = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix}$

   To solve this case, prove that any eigenvector $v = (v_1, v_2, v_3)$ is either even or odd with respect to the transformation $v_1 \leftrightarrow v_3$, $v_2 \leftrightarrow v_2$.

   (d) $M = \begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 3 & 1 & 1 & 2 \end{pmatrix}$

   A trick similar to that in (c) is needed also in this case.

2. Consider the $N \times N$ matrix

$$M = \begin{pmatrix} 0 & \Delta & 0 & \cdots & 0 & 0 & \Delta \\ \Delta & 0 & \Delta & \cdots & 0 & 0 & 0 \\ 0 & \Delta & 0 & \cdots & 0 & 0 & 0 \\ & & & \cdots & & & \\ & & & \cdots & & & \\ 0 & 0 & 0 & \cdots & \Delta & 0 & \Delta \\ \Delta & 0 & 0 & \cdots & 0 & \Delta & 0 \end{pmatrix}$$

   in which the only nonzero elements are the elements one step off the diagonal, and the elements M_{1N} and M_{N1}, and all of these elements are equal to Δ, with $\Delta > 0$.

   (a) Show that the vector

$$v = (1, 1, \cdots, 1)$$

   is an eigenvector of M.

(b) Show that the vector

$$v = (1, e^{2\pi i/N}, e^{4\pi i/N}, \cdots, e^{2(N-1)\pi i/N})$$

is an eigenvector of M.

(c) Show that every vector of the form

$$v_m = (1, e^{2\pi im/N}, e^{4\pi im/N}, \cdots, e^{2(N-1)\pi im/N})$$

where m is an integer, is an eigenvector of M. Show that these give exactly N independent eigenvectors.

(d) Show that the maximum eigenvalue is $\lambda = 2\Delta$, found already in (a). What values of m in (c) give eigenvalues close to this maximum?

(e) The Schrödinger equation for a free particle is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$$

Consider this equation with periodic boundary conditions

$$\psi(x+L) = \psi(x)$$

Find the eigenfunctions and eigenvalues.

(f) A discrete version of this equation, on a lattice labelled by integers, is

$$-\frac{\hbar^2}{2ma^2}(\psi_{n+1} - 2\psi_n + \psi_{n-1}) = E\psi_n$$

where a is the lattice spacing. We can impose periodic boundary conditions

$$\psi_{n+N} = \psi_n$$

and associate $Na = L$ to relate this to the problem of part (e). Write this equation as a matrix eigenvalue problem. Solve this problem, using the eigenvectors of M above. Relate the solution to the solution of part (e).

3. Our theory of orthogonal functions tells us that any function on the interval $[-1, 1]$ can be written as a linear combination of Legendre polynomials

$$f(x) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(x)$$

where

$$c_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} dx\, P_\ell(x)\; f(x)$$

For the following functions, work out the first 3 or 4 terms of this series (hopefully, letting MatLab or Mathematica do the integrals and sums), and graph the exact function and the successive approximations.

(a) $f(x) = 3x^2 - 4x + 2$. (This is very easy, but can you see that c_ℓ will be zero for all $\ell > 2$?)

(b) $f(x) = \sin(\pi x)$

(c) $f(x) = \exp[-ax^2/2]$. Start with a = 1. What happens when a becomes large?