Physics 130– Problem Set # 3

(due Wednesday, January 30)

1. This problem reviews properties of the Hydrogen atom wavefunctions.

   (a) In Table 4.7 on p. 154, Griffiths gives the radial eigenfunctions of the Hydrogen atom for $n = 1, 2, 3, 4$. For all of the functions up through $n = 3$, verify that these functions are normalized to the convention

   $$\int_0^\infty dr\, r^2\, |R_{n\ell}(r)|^2 = 1$$

   and that functions with the same value of ℓ are orthogonal according to

   $$\int_0^\infty dr\, r^2\, R_{n\ell}\, R_{n'\ell} = 0 \quad \text{for} \quad n \neq n'$$

   (b) For each of these eigenfunctions, through $n = 3$, compute $\langle r\rangle$. Notice that, for given n, $\langle r\rangle$ is *largest* at $\ell = 0$ and smaller for higher ℓ. Why?

2. The rotationally symmetric three-dimensional harmonic oscillator has the potential

   $$V(x, y, z) = \frac{1}{2} m\Omega^2 (x^2 + y^2 + z^2)$$

   We can solve this problem in two ways. On one hand, since the problem is rotationally symmetric, we know that the eigenfunctions can be written in the form

   $$\psi(r, \theta, \phi) = F_{n\ell}(r) Y_{\ell m}(\theta, \phi)$$

   On the other hand, the problem is equivalent to three independent harmonic oscillators. The second point of view is easiest, because we already have the complete solution of the harmonic oscillator problem.

   (a) Find the lowest energy eigenstate of the three-dimensional harmonic oscillator. Show that its energy is $\frac{3}{2}\hbar\Omega$.

   (b) Find explicitly three energy eigenstates with energy $\frac{5}{2}\hbar\Omega$. Find linear combinations of these states that these can be written in the form

   $$F_{11}(r) Y_{1m}(\theta, \phi)$$

   (c) Show that all energy eigenfunctions of this oscillator are of the form $(n + \frac{3}{2})\hbar\Omega$, where $n = 0, 1, 2, \ldots$.

(d) Show that there are 6 independent eigenfunctions with $n = 2$. Show that 5 of these can be rearranged into linear combinations that have the form

$$F_{21}(r)Y_{2m}(\theta, \phi)$$

for $m = -2, -1, 0, 1, 2$. Identify the sixth eigenfunction in terms of a function of r times a spherical harmonic.

(e) How many eigenfunctions are there with $n = 3$? How can they be grouped into sets of eigenfunctions with definite ℓ? What values of ℓ appear?

(f) How many eigenfunctions are there with $n = n$?

(g) Show that the eigenfunctions at $n = n$ always include one set of states with $\ell = n$. Show that, if n is even, only even ℓ states appear, and similarly for odd n.

(h) Compare the lowest energy levels to those of the Hydrogen atom. For the Hydrogen atom, the states $n = 2, \ell = 0$ and $n = 2, \ell = 1$ have the same energy. For the harmonic oscillator, the second $\ell = 0$ state has higher energy than the first $\ell = 1$ state. Explain the difference. Locate the third $\ell = 0$ state, the second $\ell = 1$ states, and the first $\ell = 2$ states. How are they related in energy? Is there a consistent pattern?

3. The proton is a composite state and has a finite size. We might think of it roughly as a sphere of radius 10^{-15} m. Similarly, an atomic nucleus is roughly a sphere with its charge Z distributed within a radius of size roughly equal to

$$Z^{1/3} \cdot 1.6 \times 10^{-15} \text{ m}$$

(a) Compute the probability that an electron in an eigenstate of the Hydrogen atom is actually found inside the proton, for each of the eigenstates through $n = 3$. You may assume that the deviation of the proton charge distribution from a point charge does not affect the shape of the wavefunction, most of which lies at much larger radii. One significant figure accuracy is sufficient.

(b) For a nucleus of charge Z, how do the probabilities found in part (a) vary with Z?

(c) Compute this probability numerically for the innermost electron ($n = 1$) in iron and lead atoms. (Actually, there are two innermost electrons, so the probability is doubled.)

(d) The muon μ^- is a particle identical to the electron except that its mass is 207. times heavier. A muonic atom is a bound state of a muon and an atomic nucleus. Compute the probability that a muon in the $n = 1$ state is inside the nucleus, for H, Fe, and Pb.