Physics 130– Problem Set # 2

(due Wednesday, January 23)

1. This problem develops some properties of the Hermite polynomials and the quantum harmonic oscillator wavefunctions.

   (a) The Hermite differential equation is

   $$\frac{d^2}{d\xi^2}H_n(\xi) - 2\xi\frac{d}{d\xi}H_n(\xi) + 2nH_n(\xi) = 0$$

   Find a solution to this equation for $n = 2$ of the form

   $$H_2(\xi) = 4\xi^2 + a_2$$

   That is, solve for a_2 to solve the equation. Find a solution for $n = 3$ of the form

   $$H_3(\xi) = 8\xi^3 + a_3\xi$$

   Find a solution for $n = 4$ of the form

   $$H_4(\xi) = 16\xi^4 + a_4\xi^2 + b_4$$

   (b) Show that the *generating function* $S(\xi, s)$

   $$S(\xi, s) = e^{\xi^2}e^{-(\xi-s)^2}$$

   satisfies

   $$\frac{\partial^2}{\partial\xi^2}S(\xi, s) - 2\xi\frac{\partial}{\partial\xi}S(\xi, s) + 2s\frac{\partial}{\partial s}S(\xi, s) = 0$$

   (c) Expand this equation in powers of s by plugging in the series

   $$S(\xi, s) = \sum_{n=0}^{\infty}\frac{s^n}{n!}H_n(\xi)$$

   I claim that I have correctly associated the series coefficients with the Hermite polynomials. Demonstrate this by showing that (1) the coefficient $H_n(\xi)$ is a polynomial, (2) the nth polynomial is of the form

   $$H_n(\xi) = 2^n\xi^n + \cdots$$

   (3) that the nth coefficient satisfies the Hermite equation with eigenvalue n, and (4) this approach reproduces the results of part (a).

(d) Evaluate the Gaussian integral

$$\int_{-\infty}^{\infty} d\xi \ e^{-\xi^2} S(\xi, s) S(\xi, t)$$

treating s and t as independent variables. Expand the result in s and t, and show that it implies the orthogonality of the Hermite polynomials. The coefficient of the $s^n t^n$ term gives the normalization of the Hermite polynomials. Use this result to write the correctly normalized harmonic oscillator wavefunctions.

2. In class, we sketched the solution of the Schrödinger equation for a cylindrical square well potential, that is, a potential in 2 dimensions of the form

$$V(r) = \begin{cases} 0 & r < a \\ \infty & r > a \end{cases}$$

We showed that the eigenfunctions are of the form

$$\psi_{jm}(r, \phi) = N_{jm} e^{im\phi} J_m(k_{jm} r)$$

where $J_m(z)$ is a *Bessel fuction*,

$$k_{jm} = [2mE_{jm}/\hbar^2]^{1/2}$$

E_{jm} is the jth eigenvalue for given m, and N_{jm} is a normalization constant. In this problem, you can derive some properties of Bessel functions and use them to complete the solution.

(a) The Bessel function $J_m(z)$ is the regular solution of Bessel's equation

$$\left[z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} + (z^2 - m^2) \right] J_m(z) = 0$$

We could consider this equation for any real m, but here m will be an integer $m = 0, 1, 2, \ldots$. Plug the Taylor expansion

$$J_m(z) = (\frac{1}{2} z)^p (1 + a_m z^2 + b_m z^4 + \cdots)$$

into Bessel's equation. Show that $p = m$ and find the coefficients a_m and b_m. Check your results against the Digital Library of Mathematical Functions (DLMF)

http//dlmf.nist.gov

Section 10.2.

(b) Define a function $f(z)$ by

$$J_m(z) = f(z)/\sqrt{(z)}$$

Plug this into Bessel's equation and show that $f(z)$ satisfies a Schrödinger equation with potential that goes to zero as $z \to \infty$. Argue that, for large values of z,

$$J_m(z) \sim A_m \frac{\cos(z + \phi_m)}{\sqrt{z}}$$

It is not so easy to compute A_m and ϕ_m; that is for the advanced course. The complete expression for $z \to \infty$ is given in the DLMF, section 10.7.

(c) Using your intuition about the Schrödinger equation, sketch the form of $J_0(z)$ and $J_1(z)$ for $z > 0$. Compare your results to the figures at Wolfram Mathworld

http://mathworld.wolfram.com/BesselFunctionZeros.html

This web page also gives a handy table of the zeros of $J_m(z)$.

(d) Using the table, find the energies of the three eigenfunctions of the Schrödinger equation with lowest energy for the cylindrical square well for each of $m = 0, 1, 2, 3$.

(e) Scanning over values of m, find the lowest 6 energy eigenvalues for this problem. What values of m give these eigenvalues?

3. Another Schrödinger problem that is readily solved by knowledge of special functions is the linear potential in 1 dimension

$$V(x) = k|x|$$

This is a symmetrical potential, so the eigenfunctions will be even and odd functions. We can find these by working in the region $x > 0$, where $V(x) = kx$. The odd functions will vanish at $x = 0$, the even functions will have vanishing first derivative at $x = 0$.

(a) Using intuition about the Schrödinger equation, sketch the 3 eigenfunctions of lowest energy.

(b) Write the time-independent Schrödinger equation. By appropriate rescaling and shift of x

$$x = c(z - a)$$

show that this equation can be written in the form

$$\frac{d^2}{dz^2}\psi(z) = z\ \psi(z)$$

This is Airy's differential equation.

(c) The solutions of this equation are *Airy functions* $Ai(z)$ and $Bi(z)$. You can find graphs of these functions at http://mathworld.wolfram.com/AiryFunctionZeros.html

This web pages also gives the zeros of the Airy functions. It does not give the points where the first derivative of $Ai(z)$ vanishes. However, you can read these off from the figure with sufficient accuracy to do the rest of this problem. Make a table of the first 3 such points.

(d) Look up the properties of the Airy functions as $z \to \infty$ at MathWorld or the DLMF and show that you must use only $Ai(z)$ to obtain correct boundary conditions at infinity.

(e) Solve for the first three even eigenfunctions of the Schrödinger equation and the corresponding energy eigenvalues. This will require the table you made in (c). Do not worry about normalizing these functions.

(f) Solve for the first three odd eigenfunctions of the Schrödinger equation and the corresponding energy eigenvalues. Show that these interleave the eigenvalues from (e).