Physics 130– Problem Set # 1

(due Wednesday, January 16)

1. This problem reviews some standard probability distributions that will arise in this course. A probability distribution $p(x)$ in one variable gives the probability ΔP that the value of x lies in a small interval of size Δx about $x = x_*$ as

$$\Delta P = p(x)\Delta x$$

Probability distributions in spaces of higher dimension can be defined in a similar way. We define expectation values in such a probability distribution as

$$\langle A(x) \rangle = \int dx\, p(x) A(x)$$

For example, the expectation value of x itself, the *mean* value of x, is

$$\mu = \langle x \rangle = \int dx\, p(x)\, x\ .$$

The *variance* of the distribution is

$$\sigma^2 = \left\langle (x-\mu)^2 \right\rangle\ .$$

(a) It must be true that

$$\int dx\ p(x) = 1$$

when the integral extends over all values of x. Why?

(b) Show that

$$\sigma^2 = \left\langle x^2 \right\rangle - (\langle x \rangle)^2$$

(c) Consider the probability distribution that is constant over an interval

$$p(x) = 1 \text{ for } x \in (0,1)$$

Compute the mean and the variance.

(d) Imagine that you have constructed a particle detector with strips of width ℓ, with centers at $x = n\ell$ for n an integer. The counter connected to a strip fires when a particle goes through that particular strip. The position of the hit in strip n, with standard error, would then be quoted as $n\ell \pm \sigma\ell$, where σ is the number computed in (c). Does it make sense that the quoted error $\sigma\ell$ is less than half the width of the strip?

(e) Consider the exponential probability distribution

$$p(t) = Ae^{-t/\tau} \text{ for } t > 0$$

Find A such that this distribution is properly normalized to respect (a). Compute the mean and variance of t. The *half-life* of this distribution is the time $t_{1/2}$ such that the probability that $t < t_{1/2}$ is $\frac{1}{2}$. What is the relation between the half-life and the mean value of t?

(f) Consider the Gaussian probability distribution

$$p(x) = B \exp[-(x-a)^2/2\sigma^2] \text{ for } x \in (-\infty, \infty)$$

Compute the appropriate value of the constant B, using the result

$$\int_{-\infty}^{\infty} dx\ e^{-x^2} = \sqrt{\pi}$$

Compute the mean of the distribution.

(g) It is very useful to be able to compute arbitrary moments of a Gaussian distribution, $\langle x^n \rangle$ for any n. As a starting point, consider the integral

$$\int_{-\infty}^{\infty} dx \exp[-\alpha x^2/2]\, x^n$$

show that this integral is zero for all odd n. For even n, you can develop the general formula by noting that

$$\frac{-\partial}{\partial \alpha} \int dx \exp[-\alpha x^2/2] = \int dx \exp[-\alpha x^2/2] \cdot (\frac{x^2}{2})$$

Work out the first few cases until you see the pattern. Then write down the general formula for the integral for arbitrary even n.

(h) With this in hand, compute the variance of the distribution in (f)–which should be σ^2–and also compute $\langle x^3 \rangle$ and $\langle x^4 \rangle$.

2. Solve for the discrete eigenvaues and eigenfunctions of the Schrödinger equation for a square well of finite depth

$$V(x) = \begin{cases} -W & |x| < a/2 \\ 0 & |x| > a/2 \end{cases}$$

This problem is treated in Griffiths, section 2.6, but the analysis is not hard, and you should be able to solve the problem without consulting Griffiths.

(a) Let $E = -\epsilon$ be a discrete state energy. Argue that $\epsilon < W$. Define

$$\ell = [2m(W-\epsilon)/\hbar^2]^{1/2}$$

(b) Find the most general solution of the Schrödinger equation for $|x| < a/2$. Find the most general solution of the Schrödinger equation with sensible behavior as $|x| \to \infty$ in each of the regions $x > a/2$, $x < -a/2$.

(c) Now we must match these solutions at the boundaries $x = a/2$ and $x = -a/2$. I claim that the wavefunction and its first derivative must be continuous at these boundaries. Why?

(d) We proved in class that eigenfunctions of the Schrödinger equation for a reflection-symmetrical potential $V(x) = V(-x)$ are either even or odd under reflection. First solve for the even solutions. Show that this requirement fixes the funtional form of the solution in each region. The only freedom in each region is an overall constant.

(e) Write the equations for the matching across $x = a/2$. Combine the equations, gathering the trig functions onto the right-hand side of the equation. Notice that the unknown constants drop out, and we obtain an equation for ϵ.

(f) This equation for ϵ is nicely solved graphically: Plot the left- and right-hand sides of the equation as functions of ℓ, and see where the curves intersect. For the case

$$W = \frac{9\hbar^2\pi^2}{2ma^2}$$

(the energy of the third discrete state of the infinite square well) plot the curves carefully and find the energies accurately (in units of $\hbar^2\pi^2/2ma^2$).

(g) Show that there is always a discrete eigenstate for any $W > 0$.

(h) Show that each even eigenstate lies at an energy below that of the corresponding energy eigenstate of the infinite square well.

(i) Now turn to the case of odd eigenfunctions. Analyze this case in parallel with the even case. Show that the W given in (f) has one odd discrete state, and find its energy accurately.

(j) Find the relation of the energies of the odd eigenfunctions to the corresponding discrete energies of the infinite square well.

(k) Sketch the eigenfunctions for W given in (f). Show that the zeros of each successive eigenfunction interleave those of the previous one.