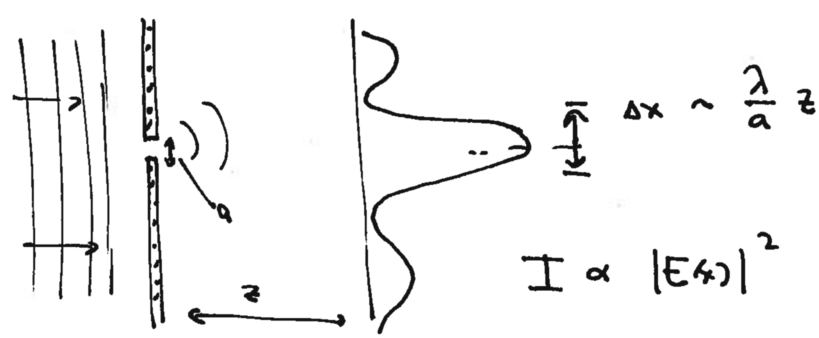
More about Quantum Coherence

In this lecture, I will present more examples illustrating the idea of *quantum coherence.* I will discuss a canonical example of coherence in quantum mechanics. Then I will present a system in which quantum coherence creates fascinating phenomena seen over macroscopic length scales. Finally, I will discuss what might be the ultimate example of quantum coherence and find a new picture of quantum-mechanical evolution.

Richard Feynman begins Volume III of the Feynman Lectures with two lectures that explain the ideas of quantum coherence and the collapse of the wavefunction in the context of an experiment involving the diffraction of electron waves. I encourage you to read the whole discussion, but I will make a precis of it here. I emphasize that Feynman discusses a thought experiment. However, the major consequences of this experiment have been verified in the laboratory for electrons, photons, and neutrons using somewhat more complex experimental setups.

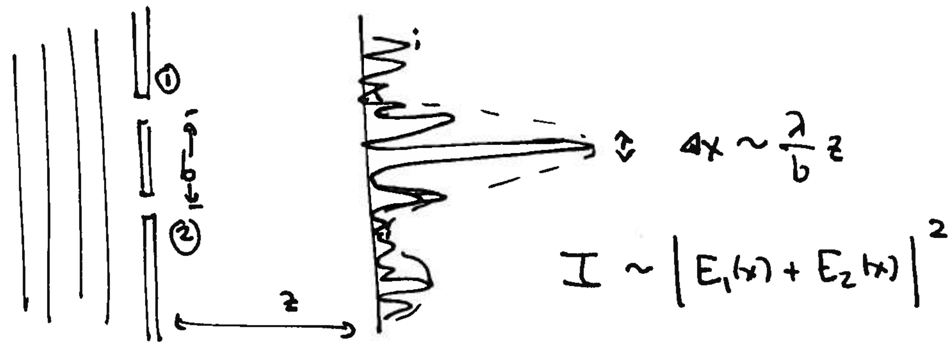
Consider, then, an experiment in which electrons are prepared in states that are approximately plane waves. The waves impinge on an absorber with small holes in it. Behind the absorber, there is a phosphorescent screen. The wave pattern on this screen is the diffraction pattern created by the holes.

If there is one hole, we have the single-slit diffraction pattern. In an experiment with classical diffraction, the wave impinging on trhe absorber would be a macroscopic light wave. The light intensity on the screen would be proportional to the square of the electric field.



For individual electrons or photons, the role of the electric field is played by the Schrödinger wavefunction. Each electron creates one flash of light on the phosphorescent screen. The square of the Schrödinger wavefunction $|\psi(x)|^2$ gives the probability that the spot of light occurs in at a location x. This probability can be measured by repeating the experiment many times.

Next, consider the case in which there are two holes in the absorber. The wave amplitude on the screen is now a 2-slit diffraction pattern



For electrons, we still see invidual flashes, but the pattern of the flashes is changed, with many additional zeros and a much narrower central peak. The single electron goes through *both* slits. The probability of the position of the flash is given by summing the waveforms from the two slits, and then squaring to obtain the probability.

$$\text{Prob.} \propto |\psi_1(x) + \psi_2(x)|^2$$

This is the prescription associated with two *coherent* contributions. More generally, for coherent processes, we add the amplitudes and then square the result to obtain the probabilities.

We could modify the experiment to attempt to identify which hole the electron went through. For example, we could install a detector in each hole that lets the electron pass through but registers a count if the electron passes through that hole. However, the registration of the counter involves a macroscopic change in the state and, in according to the ideas in the previous lecture, collapses the wavefunction. Then the electron that goes to the screen is in the waveform generated by one hole only. You might say that we could choose not to look at the counters. Still, the contributions from the two holes would add *incoherently*. In that case, the probability distribution of flashes on the screen would be

$$\text{Prob.} \propto |\psi_1(x)|^2 + |\psi_2(x)|^2$$

More generally, for incoherent contributions, we square the wavefunctions separately and then add these contributions with the appropriate probabilities.

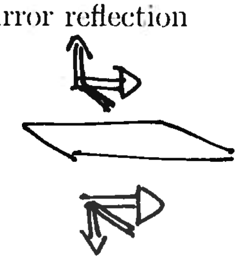
Keeping these rules in mind, I would now like to analyze a system in elementary particle physics that provides a very beautiful example of quantum coherence. This system also provides a setting for understanding some of the fundamental symmetries of nature.

In our discussion of the rotation group, I remarked that there is a part of this group called $SO(3)$ that is infinitesimally generated. The generators, the operators J^i are good symmetries of nature and commute with the real Hamiltonian of nature as far as we have been able to test. However, the rotation group also contains a discrete element

$$P: \ \vec{x} \to -\vec{x}$$

called *Parity* or P which cannot be realized by a series of infinitesimal transformations. Parity is related to mirror reflection; a mirror reflection

$$x\,y\,z \to (x, y, -z)$$



is the operation of parity combined with a 180° rotation about the $\hat{z}$ axis. The fact that $[\vec{J}, H] = 0$ does not imply that P commutes with H. This is a separate question that must be investigated experimentally. In atomic physics, P leads to rules forbidding many atomic transitions, and these rules are seen to be satisfied to high accuracy. There are additional discrete symmetries that are natural candidates for accurate symmetries of nature. One of these is *Time-Reversal* or T, the reversal of the direction of time evolution. Another, which we encounter in relativitisti problems, is *Charge-conjugation* or C, the interchange of particles and antiparticles. In atomic physics and in quantum electrodynamics, P, C, and T all commute with the Hamiltonian. In relativistic quantum field theory, it is possible to prove a theorem that the operator PCT, which simultaneously implements the three discrete symmetries, must always commute with the Hamiltonian.

Now let me set up the description of the system that I wish to study. The lightest elementary particles interacting through the strong nuclear forces are called π mesons. There are three π mesons, all with $mc^2 \approx 135$ MeV. These are $J = 0$ bound states of a quark and an antiquark,

$$\pi^+ = (u\bar{d}) \qquad \pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \qquad \pi^- = (d\bar{u})$$

where u and d denote types of quark. In the relativistic theory of spin $\frac{1}{2}$ particles, the operator P acting on an antiquark wavefunction gives an extra factor (-1). We say that antifermions have *intrinsic parity* (-1).

$$P|q\rangle = +|q\rangle \qquad P|\bar{q}\rangle = (-1)|\bar{q}\rangle$$

Then the π mesons also have intrinsic parity $P = -1$,

$$P|\pi^a\rangle = (-1) \cdot |\pi^a\rangle$$

This can be checked experimentally. For example, the π^0 decays to two gamma ray photons. In a state with $P = +1$, the two photon polarizations would be required to be parallel, but these polarizations are observed to be orthogonal to one another.

There is another, heavy quark called s. The $J = 0$ bound states of s and d quarks and antiquarks are called neutral K mesons. There are two types,

$$K^0 = (d\bar{s}) \qquad \bar{K}^0 = (s\bar{d})$$

These states also have $P = -1$. The action of C interchanges these states

$$C|K^0\rangle = -|\bar{K}^0\rangle \qquad C|\bar{K}^0\rangle = -|K^0\rangle$$

The minus sign that appears here is the standard convention used in particle physics. The expectation value of H in these states must be exactly equal, as a consequence of CPT symmetry. However, there is a process that causes the transition

$$d\bar{s} \rightleftarrows s\bar{d}$$

at a very small rate. This situation is similar to the one that we met in our study of 2-level systems. The eigenstates of the Hamiltonian are the mixtures

$$|K_S^0\rangle = \frac{1}{\sqrt{2}}\left[|K^0\rangle + |\bar{K}^0\rangle\right]$$

$$|K_L^0\rangle = \frac{1}{\sqrt{2}}\left[|K^0\rangle - |\bar{K}^0\rangle\right]$$

I will explain the labels S and L in a moment. These states have $P = -1$ and are also eigenstates of C, with

$$C\,|K_S^0\rangle = (-1)\,|K_S^0\rangle \qquad C\,|K_L^0\rangle = (+1)\,|K_L^0\rangle$$

The masses of the neutral K mesons are $mc^2 = 498$ MeV, with a small mass difference of

$$\Delta m c^2 = 3.484 \times 10^{-12}\ \text{MeV}$$

Note that

$$\frac{\hbar}{\Delta m c^2} = 1.9 \times 10^{-10}\ \text{sec} \qquad \frac{\hbar c}{\Delta m c^2} = 5.7\ \text{cm}$$

so the beat frequency corresponds to macroscopic times and distances.

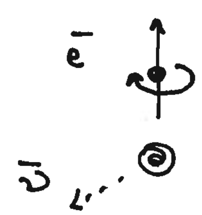
The neutral K mesons are unstable. They decay to states with 2 or more π mesons through the *weak interactions* that mediate radioactive decay of nuclei. The products dominantly are states with zero orbital angular momentum. In this case

$$P\,|2\pi\rangle = (+1)\,|2\pi\rangle \qquad\qquad C\,|2\pi\rangle = (+1)\,|2\pi\rangle$$

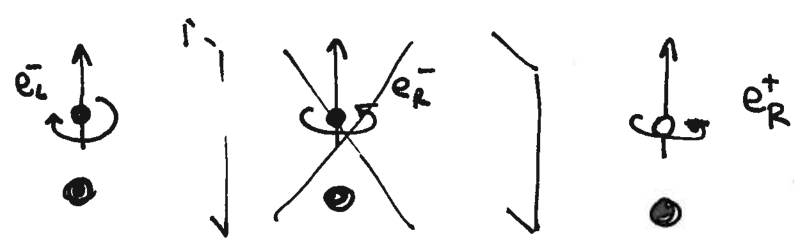
$$P\,|3\pi\rangle = (-1)\,|3\pi\rangle \qquad\qquad C\,|3\pi\rangle = (+1)\,|3\pi\rangle$$

You can see that, if both P and C commute with H, the decay of K_L^0 to 3π, is allowed but the K_S^0 is forbidden to decay either to the 2π state, which has a different value of P or to the 3π state, which has a different value of C.

However, the K_S^0 is actually observed to decay to 2π. This was the first evidence that the weak interactions do not respect the symmetry P. Today, we have much stronger and more precise evidence of this. In particuar, when a nucleus emits an electron in β decay, that electron is polarized in the direction of left-handed spin.



The mirror reflection of this state is a different state, in which the electron spins in the right-handed sense. However, it is observed that positrons emitted in β^+ decays are polarized with right-handed spin.



So in nuclear β decay, the combined operator CP is apparently a symmetry of the Hamiltonian.

Notice that if P and C do not separately commute with the Hamiltonian but CP is a symmetry, then

$$CP\,|K_S^0\rangle = (+1)\,|K_S^0\rangle \qquad CP\,|K_L^0\rangle = (-1)\,|K_L^0\rangle$$

$$CP\,|2\pi\rangle = (+1)\,|2\pi\rangle \qquad CP\,|3\pi\rangle = (-1)\,|3\pi\rangle$$

and the decays $K_S^0 \to 2\pi$, $K_L^0 \to 3\pi$ are allowed. The decay to 3π is hindered by the fact that three π mesons add to an energy very close to the mass of the K meson. So the decay to 2π is relatively fast (though still slow compared to the rate of a strong interaction process), while the decay to 3π is slow. The measured lifetimes are

$$\tau(K_S^0) = 0.896 \times 10^{-10}\,\mathrm{sec} \qquad \tau(K_L^0) = 5.12 \times 10^{-8}\,\mathrm{sec}$$

$$c\tau(K_S^0) = 2.7\,\mathrm{cm} \qquad c\tau(K_L^0) = 15.3\,\mathrm{m}$$

so we call the K eigenstates the K-short (K_S^0) and K-long (K_L^0).

The decays of the K mesons are probabilistic, occuring at a fixed rate per unit time. Thus, the decay laws are exponential. We can represent this by adding a factor

$$e^{-\frac{\Gamma}{2}\frac{t}{\hbar}}$$

to the time evolution of the wavefunctions. Here Γ is an inverse decay time measured in energy units, and the decay probability distribution is

$$e^{-t/\tau} = e^{-(\Gamma/\hbar)t} \qquad \tau = \frac{\hbar}{\Gamma}$$

The wavefunction $|K_S^0\rangle$ then has the time dependence

$$|K_S^0(t)\rangle = e^{-i m_S c^2 t/\hbar}\, e^{-i\frac{\Gamma_S}{2}t/\hbar}\,|K_S^0\rangle + (2\pi \text{ decay products})$$

The decay products are high-energy particles that typically light up the surrounding material. Then these pieces of the wavefunction become incoherent with the K^0 state. However, the undecayed K^0 wavefunction maintains coherence over macroscopic distances. The time dependence of the wavefunction $|K^0_L\rangle$ has a similar form

$$|K^0_L(t)\rangle = e^{-i m_L c^2 t/\hbar}\, e^{-\Gamma_L/2\, t/\hbar}\, |K^0_L\rangle + (3\pi \text{ decay products})$$

In these expressions, t is the *proper time*, that is, the time in the rest frame of the K^0. This is related to time in the lab frame by $t_{lab} = t\gamma$, where $\gamma = E_K/m_K c^2$ is Einstein's time dilation factor.

We can make K^0 mesons through the reaction

$$\pi^-(d\bar{u}) + p(uud) \rightarrow \Lambda^0(uds) + K^0(d\bar{s})$$

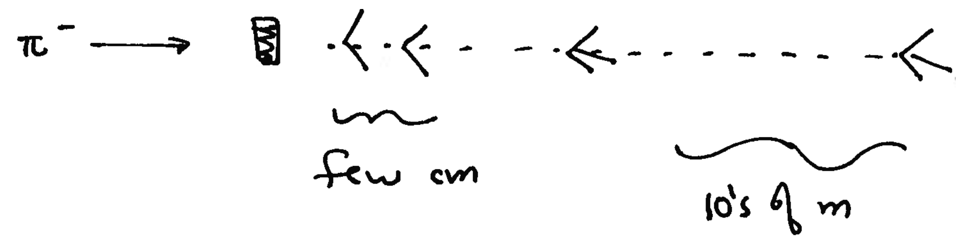
Notice that the state created here is the K^0, with the quark content $d\bar{s}$. In terms of the energy eigenstates, this is

$$|K^0\rangle = \frac{1}{\sqrt{2}}\left[|K^0_S\rangle + |K^0_L\rangle\right]$$

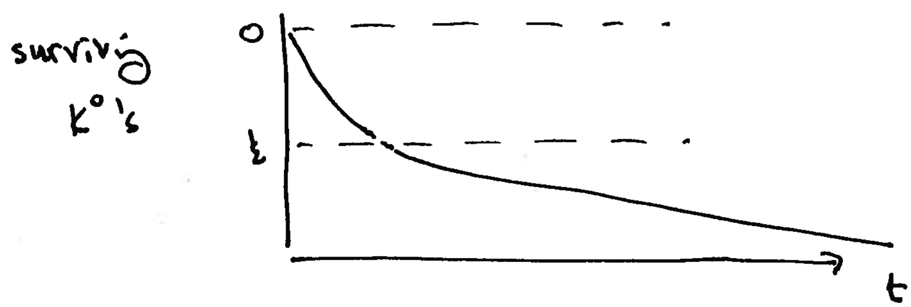
The time evolution of this state is

$$|K^0\rangle(t) = \frac{1}{\sqrt{2}}\left[e^{-i(m_S c^2 - i\Gamma_S/2)\frac{t}{\hbar}}\, |K^0_S\rangle + e^{-i(m_L c^2 - i\frac{\Gamma_L}{2})\frac{t}{\hbar}}\, |K^0_L\rangle\right] + (\text{products})$$

Now, what do we actually observe when we collider a beam of π^- mesons with a block of material? The distribution of decay products is



That is, we see an exponential distribution of 2π decays concentrated within a few cm behind the target, and a broader distribution of 3π decays extending for tens of m. Each exponential contains half of the total number of K mesons originally produced.



Beyond a 1m or so, the K_S^0 component of the initial state is essentially gone and all that is left is a wavefunction proportional to

$$e^{-i(m_Lc^2 - \Gamma_L/2)t/\hbar} \quad |K_L^0\rangle$$

We can now put some additional material (for example, a slab of copper) in the path of the beam. What happens? The K^0 and $\overline{K}^0$ components of the wavefunction interact differently with the material. In particular, the $\overline{K}^0$, which contains the antiquark $\overline{d}$ of a quark contained in protons and neutrons, is strongly absorbed. After traversing a length L of material, the state $|K_L^0\rangle$ is transformed into

$$\frac{1}{\sqrt{2}}\left[e^{-(a+i\gamma)L} \quad |K^0\rangle \quad - \quad e^{-(b+i\delta)L} \quad |\overline{K}^0\rangle\right]$$

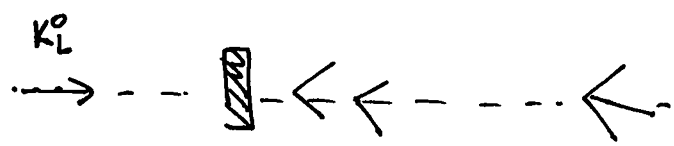
with $b > a$. We can reorganize this state into a form

$$\alpha |K_L^0\rangle + \beta |K_S^0\rangle \qquad \beta = \tfrac{1}{2}\left[e^{-(a+i\gamma)} - e^{-(b+i\delta)}\right]$$

The time-dependence of this state is

$$\alpha\, e^{-i(m_L c^2 - i\Gamma_L/\hbar)t/\hbar} |K_L^0\rangle + \beta\, e^{-i(m_S c^2 - i\Gamma_S/\hbar)t/\hbar} |K_S^0\rangle$$

So we see 2π decays again in a region of a few cm behind the slab of material, occurring with probability $|\beta|^2$.



This effect is called K_S^0 *regeneration*.

If CP were truly a good symmetry of nature, this would be the end of the story. However, when physicists studied the neutral K^0 decays more carefully, additional details were found. In 1964, Christensen, Cronin, Fitch, and Turlay [Phys. Rev. Lett. 13, 138 (1964)] carefully measured the decays in a beam of K_L^0's, in a Helium bag to minimize regeneration. They found that, with a probability of order 10^{-3}, the K_L^0 could decay to 2π. The observed 2π states were distributed nicely according the exponential decay law of the K_L^0.

If the K_L^0 can decay to 2π, there are actually two contributions to the 2π states seen behind a regenerator, the 2π states from K_S^0 and those from K_L^0. These states are produced coherently, and so we must compute the total amplitude for the decay and then square it to obtain the rate. If γ_S is the decay amplitude from a K_S^0 and γ_L is the decay amplitude from a K_L^0, then the total amplitude for the decay is

$$\beta \gamma_S\, e^{-i(m_S c^2 - i\Gamma_S/\hbar)t/\hbar} + \alpha \gamma_L\, e^{-i(m_L c^2 - i\Gamma_L/\hbar)t/\hbar}$$

The square of this expression is

$$\text{Prob} \propto |\beta\gamma_S|^2 e^{-\Gamma_S t/\hbar} + |\alpha\gamma_L|^2 e^{-\Gamma_L t/\hbar} + 2|\alpha\beta\gamma_S\gamma_L| e^{-\frac{\Gamma_S+\Gamma_L}{2}\frac{t}{\hbar}} \cos\left(\frac{(m_L-m_S)c^2 t}{\hbar} + \phi\right)$$

where ϕ is a phase factor. The last term here is a time-dependent interference. This interference effect is observed experimentally, as shown in the figure, with the oscillation frequency and exponential terms agreeing with the experimental prediction.

The K^0 and the $\overline{K}^0$ can also decay to states involving electrons or muons

$$K^0(d\bar{s}) \longrightarrow \ell^+\pi^-\nu \qquad \overline{K}^0(s\bar{d}) \longrightarrow \ell^-\pi^+\bar{\nu}$$

where $\ell = e$ or μ. Since the time-dependent state behind the regenerator is a coherent mixture of K^0 and $\overline{K}^0$, this state shows similar interference effects. If we produce K^0 mesons, from a target, the time-dependent state is

$$|K^0(t)\rangle = \frac{1}{\sqrt{2}}\left(|K_S^0(t)\rangle + |K_L^0(t)\rangle\right)$$

$$= \frac{1}{2}\left(\left(e^{-i(m_S c^2 - i\Gamma_S/2)\frac{t}{\hbar}} + e^{-i(m_L c^2 - i\Gamma_L/2)t/\hbar}\right)|K^0\rangle + \left(e^{-i(m_S c^2 - i\Gamma_S/2)t/\hbar} - e^{-i(m_L c^2 - i\frac{\Gamma_L}{2})\frac{t}{\hbar}}\right)|\overline{K}^0\rangle\right)$$

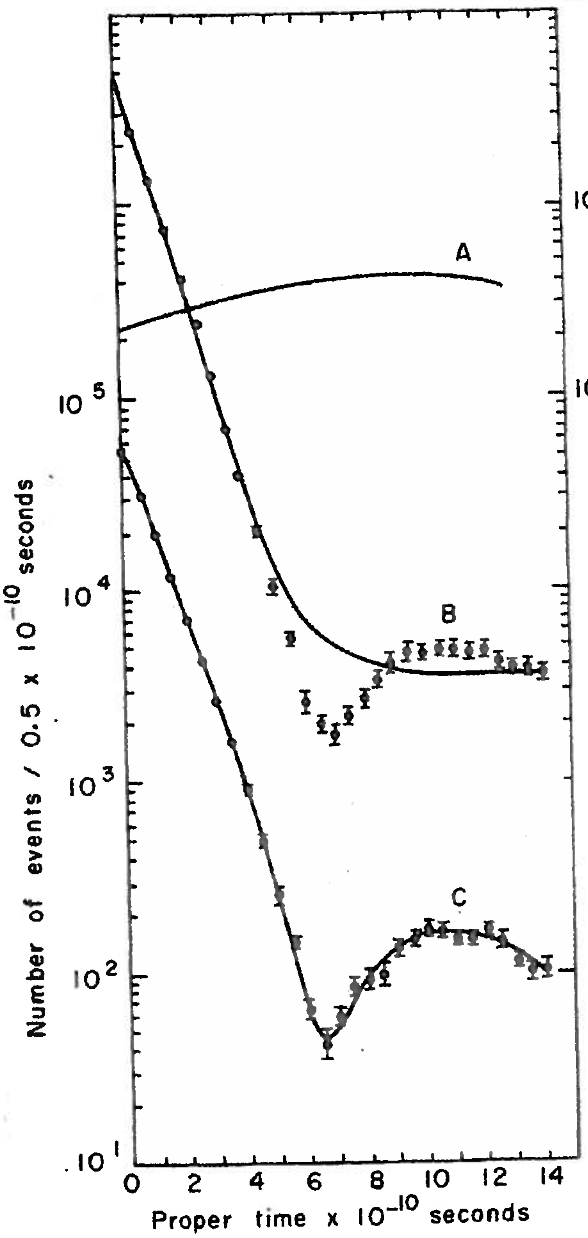
Then the number of decays to e^+ and e^- as a function of time is proportional to

$$\text{Prob}(e^+) \propto \left| e^{-i(m_L c^2 - i\Gamma_L/2)t/\hbar} + e^{-i(m_S c^2 - i\Gamma_S/2)t/\hbar} \right|^2$$

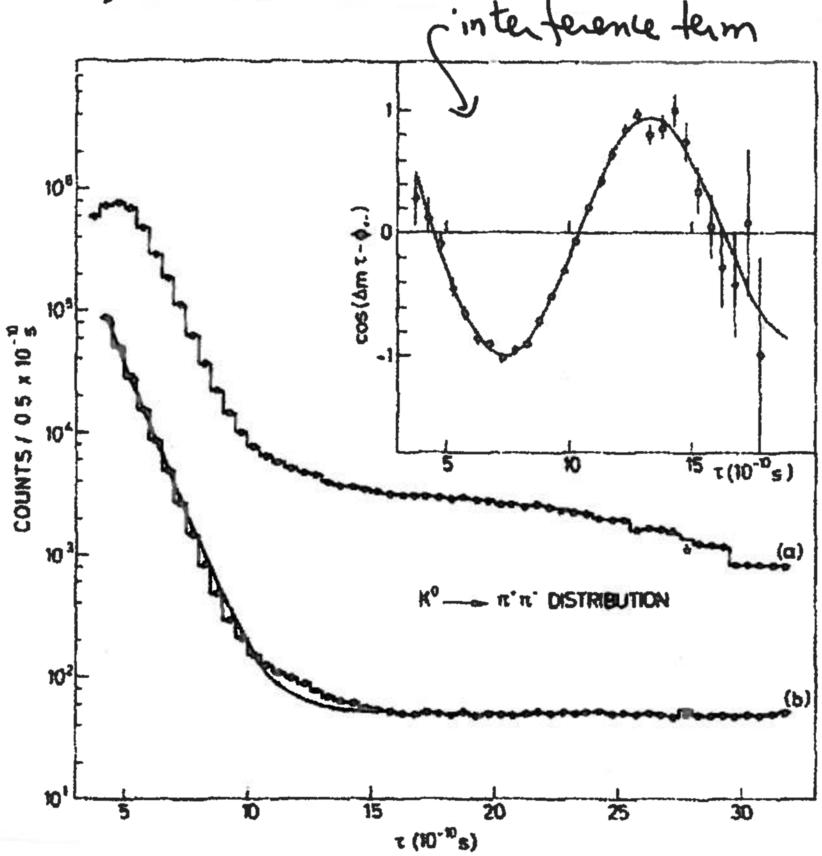
$$\text{Prob}(e^-) \propto \left| e^{-i(m_L c^2 - i\Gamma_L/2)t/\hbar} - e^{-i(m_S c^2 - i\Gamma_S/2)t/\hbar} \right|^2$$

The asymmetry

a)



b.)



Time distribution of $K^0 \to \pi^+\pi^-$ decays

a) behind a Carbon regenerator, from Carithers et al Phys. Rev Lett. 34, 1244 (1975)

b.) in K^0 production, from Gewenigr et al Phys. Lett 48B 487 (1974)

$$\frac{\mathrm{Prob}(e^+) - \mathrm{Prob}(e^-)}{\mathrm{Prob}(e^+) + \mathrm{Prob}(e^-)}$$

oscillates as a function of proper time, as shown in the figure. The final small excess of e^+ over e^- decays is due to the fact that the K^0_L wavefunction is corrected from the expression given above due to terms in H that do not respect CP. This leads to a small CP asymmetry in the wavefunction.

The evidence that K^0_L decays to 2π implies CP *violation*, the statement that CP does not commute with the full Hamiltonian describing nature. If CPT is a good symmetry, as required by quantum field theory, then T must also fail to commute with H. This was later confirmed in detailed studies of K^0 decay. Very recently, the BaBar experiment at SLAC, studying the decays of mesons containing the heavy quark b, gave very direct evidence that the equations of motion of nature are not symmetric under time reversal [Phys. Rev. Lett. 109, 211801 (2012)].

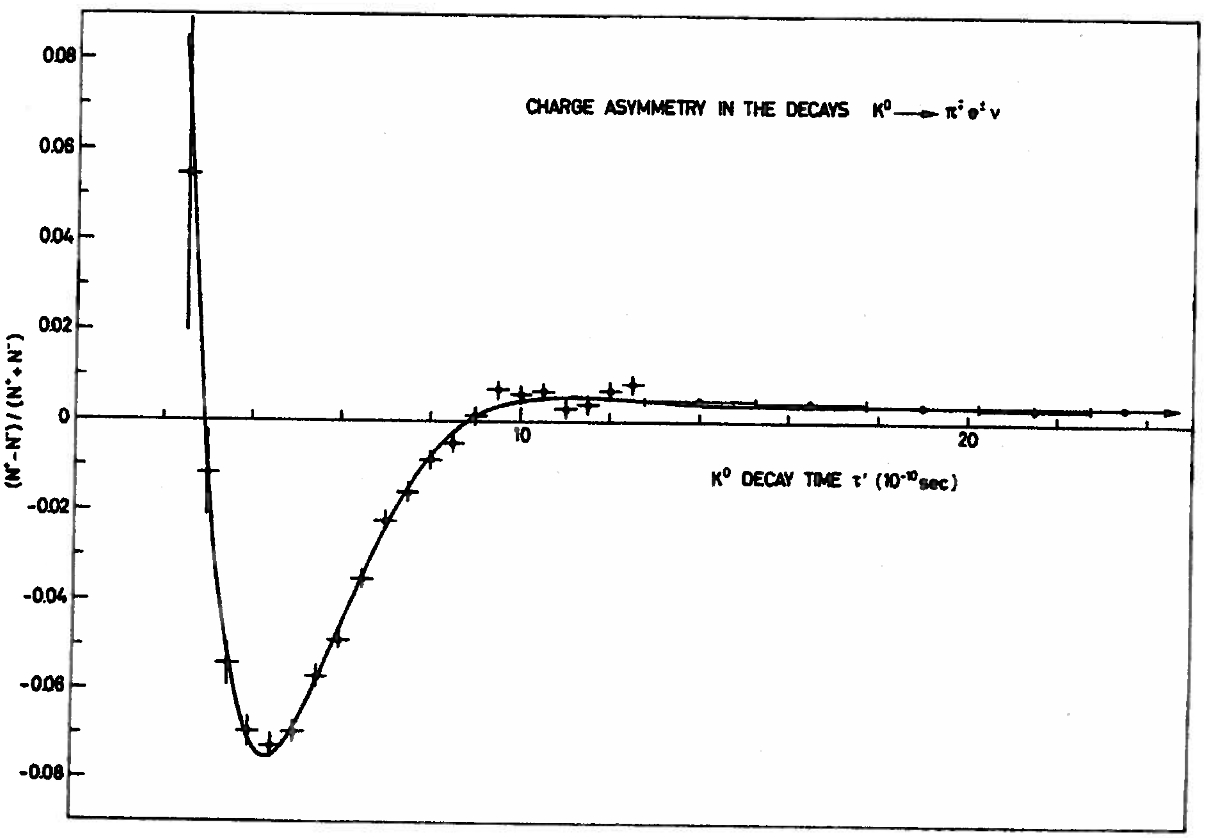
The rule that we used in analyzing these examples is that, as long as the wavefunction is coherent, we sum the amplitudes leading to a particular result and then square the sum to find the probability of the outcome of a measurement. I would now like to take this philosophy to an extreme, first enunciated by Richard Feynman. Consider the quantum motion of a particle in a potential, from the point x_0 at the initial time t_0 to the final position x_f at the later time t_f. In this discussion, I will work in one dimension for simplicity. The probability ampitude for this transition is given by

$$G(x_f, t_f | x_0 t_0) = \langle x_f | \, e^{-iH(t_f - t_0)} \, | x_0 \rangle$$

The function $G(x_f, t_f|x_0, t_0)$ is called the *propagator*. It obeys the Schrödinger equation

$$\left(i\hbar \frac{\partial}{\partial t_f} - \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_f^2} + V(x_f) \right] \right) G(x_f, t_f | x_0 t_0) = 0$$

with initial condition



Charge asymmetry of decays

$K^0 \to e^+ \pi^- \nu$ / $K^0 \to e^- \pi^+ \bar{\nu}$

as a function of the proper time since production

from Gjesdal et al Phys. Lett. 52B 113 (1974)

$$G(x_f t_f | x_0 t_0) = \delta(x_f - x_0)$$

at $t_f = t_i$. Feynman proposed that we model G as being given by a sum over all possible paths $x(t)$ by which the particle could move from x_0 to x_f. That is,

$$G(x_f t_f | x_0 t_0) = \int \mathcal{D}x(t) \; e^{+\frac{i}{\hbar} S[x(t)]}$$

where $x(t)$ is a path with $x(t_0) = x_0$, $x(t_f) = x_f$ and $\mathcal{D}x(t)$ is the integral over the space of paths. We will need to define this integral, and I will do so in a moment. In principle, each path might have its own amplitude. The complete quantum state is coherent, so we will need to add the amplitudes and then interpret the value of the propagator. These amplitudes will distinguish different quantum mechanical problems, for example, motion in a square well or motion in a harmonic oscillator potential.

Following a suggestion of Dirac, Feynman chose the amplitude for the path $x(t)$ to be a pure phase factor, as indicated above, where the phase is the classical action integral.

$$S[x(t)] = \int_{t_0}^{t_f} dt \left[\tfrac{1}{2} m \dot{x}^2 - V(x)\right]$$

divided by $\hbar$ to make it dimensionless. The classical action is naturally associated with a path. In Lagrangian classical mechanics, we choose the correct path by the principle that $S[x(t)]$ is stationary under small variations of the path.

$$\delta S[x(t)] = 0$$

This is the *principle of least action.*

I would now like to show that this model is precisely correct. To do this, I will show that this expression for G satisfies the Schrödinger equation. The correct initial condition is satisfied manifestly.

To work with the integral over paths, we must define it in a precise way. To do this, make time discrete; write $(t_f - t_0) = N\epsilon$, and represent the path by the values of $x(t)$ at the times $t_j = t_0 + j\epsilon$. Then

$$\int \mathcal{D}x(t) = \prod_{j=1}^{N-1} \left[\int_{-\infty}^{\infty} dx_j \cdot \mathcal{N} \right]$$

where x_0 is just x_0, $x_N = x_f$, and $\mathcal{N}$ is an overall constant for each integral, to be determined. In a similar way, we must replace the action integral S with a discrete approximation

$$S[x(t)] = \sum_{j=1}^{N} \epsilon \left\{ \frac{m}{2} \frac{(x_j - x_{j-1})^2}{\epsilon^2} - V\left(\frac{x_j + x_{j-1}}{2}\right) \right\}$$

Then the full expression for $G(x_f, t_f | x_0, t_0)$ is

$$G(x_f t_f | x_0 t_0) = \int dx_1 \mathcal{N} \cdots \int dx_{N-1} \mathcal{N} \; e^{\frac{i}{\hbar} \sum_{j=1}^{N} \epsilon \left\{ \frac{m}{2\epsilon^2}(x_j - x_{j-1})^2 - V\left(\frac{x_j + x_{j-1}}{2}\right) \right\}}$$

In particular, the values of the propagator at $t_N = t_f$ and at t_{N-1} are related by

$$G(x_f t_f | x_0 t_0) = \int dx_{N-1} \mathcal{N} \; e^{\frac{i}{\hbar} \frac{m (x_N - x_{N-1})^2}{\epsilon} - \frac{i}{\hbar} \epsilon V\left(\frac{x_N + x_{N-1}}{2}\right)}$$

$$(x_N = x_f)$$

We can analyze this relation for small ϵ. The factor

$$e^{\frac{im}{2\hbar}\frac{(x_f - x_{N-1})^2}{\epsilon}}$$

oscillates rapidly and kills off the integral except near $x_f - x_{N-1} = 0$. This is the desired behavior; the propagator should change slowly from one value of t_j to the next. There is a potential problem that the integral

$$\int_{-\infty}^{\infty} dx_{N-1} \; e^{\frac{im}{2\hbar}\frac{(x_f - x_{N-1})^2}{\epsilon}} = \left(\frac{2\pi\hbar\epsilon}{-im}\right)^{\frac{1}{2}}$$

introduces a new awkward factor at each time step. We can remove this by setting

$$N = \left(\frac{-im}{2\pi\hbar\epsilon}\right)^{\frac{1}{2}}$$

Then

$$G(x_f t_f \,|\, x_0 t_0) = G(x_f \; t_{N-1} \,|\, x_0 t_0) + \mathcal{O}(\varepsilon)$$

and we can expand the relation in powers of ϵ.

To go further, we need the values of the Gaussian integrals,

$$\int dy \; N \; e^{\frac{im}{2\hbar} y^2/\epsilon} = 1$$

$$\int dy \; N \cdot y \cdot e^{\frac{im}{2\hbar} y^2/\epsilon} = 0$$

$$\int dy \; N \cdot y^2 \; e^{\frac{im}{2\hbar} y^2/\epsilon} = \left(\frac{\hbar\epsilon}{-im}\right)$$

Now let $y = x_f - x_{N-1}$ and expand $G(x_{N-1}, t_{N-1}|x_0, t_0)$ in powers of y

$$G(x_f t_f | x_0 t_0) = \int dy\, N\, e^{\frac{im}{2\hbar}\frac{y^2}{\epsilon}} \Big\{ G(x_f t_f) + y \frac{\partial}{\partial x_f} G(x_f t_f) + \frac{1}{2} y^2 \frac{\partial^2}{\partial x_f^2} G(x_f t_f) - \epsilon \frac{\partial}{\partial t_f} G(x_f t_f) - i \frac{\epsilon}{\hbar} V\, G(x_f t_f) + \cdots \Big\}$$

Evaluating the integral for $G(x_{N-1}, t_{N-1}|x_0, t_0)$, we then find

$$G(x_f t_f | x_0 t_0) = G(x_f t_f | x_0 t_0) + 0 + \frac{1}{2}\left(\frac{\hbar \epsilon}{-im}\right) \frac{\partial^2}{\partial x_f^2} G(x_f t_f) - \epsilon \frac{\partial}{\partial t_f} G(x_f t_f) - i \frac{\epsilon}{\hbar} V(x_f)\, G(x_f, t_f) + O(\epsilon^2)$$

For small ϵ, we can keep only the terms of order ϵ,

$$\epsilon \frac{\partial}{\partial t_f} G(x_f, t_f) = i \frac{\hbar \epsilon}{2m} \frac{\partial^2}{\partial x_f^2} G(x_f t_f) - i \frac{\epsilon}{\hbar} V(x_f)\, G(x_f t_f)$$

and we find indeed

$$i\hbar \frac{\partial}{\partial t_f} G(x_f t_f) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_f^2} + V(x_f) \right] G(x_f t_f)$$

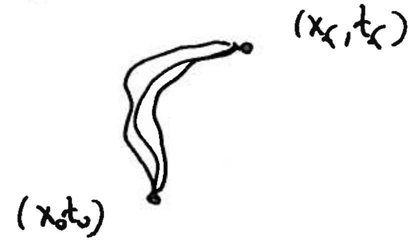
The equation

$$\langle x_f | e^{-\frac{i}{\hbar} H (t_f - t_0)} | x_0 \rangle = \int_{(x_0 t_0)}^{(x_f t_f)} \mathcal{D}x(t)\; e^{\frac{i}{\hbar} S[x(t)]}$$

gives an alternative way to express the time-dependence of a wavefunction. We might call this the *Feynman picture*. The particle or system propagates along all possible paths, accumulating along each path a phase factor equal to the classical action. The paths are then summed coherently by to obtain the complete transition amplitude.

For small $\hbar$ and macroscopic motions, the phases corresponding to nearby paths will typically be very different, and so the contributions from these paths will cancel in the integral over paths. The contributions from neighboring paths will add constructively only in regions where the action integral is unchanged to first order between neighboring paths. This is exactly the criterion

$$\delta S[x(t)] = 0$$



of Lagrangian mechanics cited above. The principle of least action arises from Feynman's formalism as the natural macroscopic approximation to the coherent sum over paths.