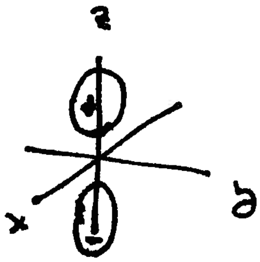
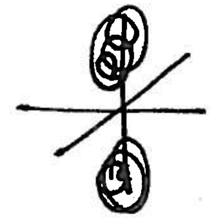
Physics 130 Midterm Exam Solutions

1. a) ψ :



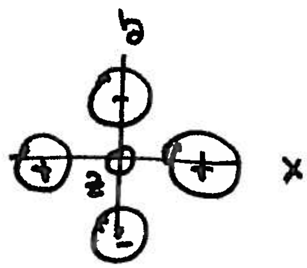
$|\psi|^2$



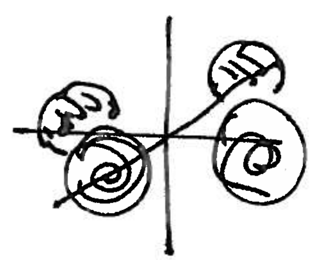
b.) the ϕ dependence is $e^{2i\phi} + e^{-i2\phi} = 2\cos 2\phi$

from above

ψ

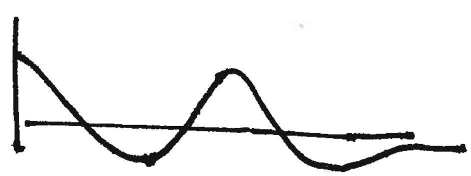


$|\psi|^2$

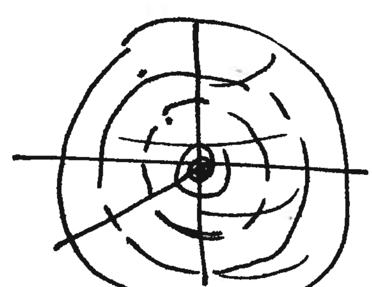


c.) The radial wavefunct. is

R_{45}



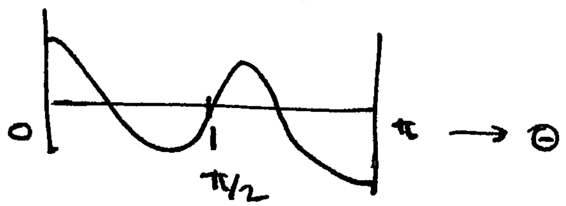
$|\psi|^2$



4 concentric spherical shells

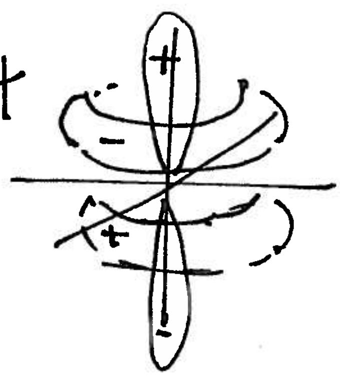
d.) The radial wavefunction of the 4F state has no zeros except for the one at $r = 0$

The Θ wavefunction has the form ($Y_{30}(\theta,\phi)$)

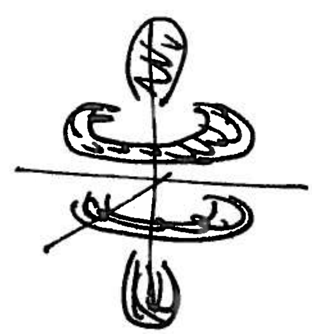


so

ψ



$|\psi|^2$



2.) a.) $\left(-(1-x^2)\frac{d^2}{dx^2} + x\frac{d}{dx}\right) 1 = 0 = 0 \cdot 1$

$\left(-(1-x^2)\frac{d^2}{dx^2} + x\frac{d}{dx}\right) x = x = 1 \cdot x$

b.) If $P_n(x)$ is a polynomial of order n: $P_n(x) = x^n + \dots$

then $\left(-(1-x^2)\frac{d^2}{dx^2} + x\frac{d}{dx}\right) P_n(x)$ is also a polynomial of degree $\underline{n}$.

Matching the terms of degree n:

$$\left(-(1-x^2)\frac{d^2}{dx^2} + x\frac{d}{dx}\right)(x^n + \dots)$$

$$= n(n-1)x^n + nx^n + \Theta(x^{n-2})$$

$$= n^2 x^n + \Theta(x^{n-2})$$

so $\lambda = n^2$

c.) $\left(-(1-x^2)\frac{d^2}{dx^2} + x\frac{d}{dx}\right)(x^2+a)$

$$= -(1-x^2)\cdot 2 + x\cdot 2x$$

$$= 4x^2 - 2 \overset{\text{should}}{\underset{\text{be}}{=}} 4(x^2+a)$$

so $a = {}^{2}\!/_{4} = \frac{1}{2}$ $\boxed{T_2(x) = x^2 - \frac{1}{2}}$

$$\left(-(1-x^2)\frac{d^2}{dx^2} + x\frac{d}{dx}\right)(x^3+ax)$$

$$= -(1-x^2)\,6x + x\cdot(3x^2+a)$$

$$= 9x^3 - (6-a)x \overset{\text{should}}{\underset{\text{be}}{=}} 9(x^3+ax)$$

so $-6 = 8a$

$a = -\frac{3}{4}$ $\boxed{T_3 = x^3 - \frac{3}{4}x}$

d.)

$$-(1-x^2)\frac{d^2}{dx^2} + x\frac{d}{dx}$$

$$= -\sqrt{1-x^2}\,\frac{d}{dx}\,\sqrt{1-x^2}\,\frac{d}{dx}$$

then

$$\int_{-1}^{1} dx\,\frac{1}{\sqrt{1-x^2}}\left(\left[-(1-x^2)\frac{d^2}{dx^2} + x\frac{d}{dx}\right]T_n(x)\right)\left(T_m(x)\right)$$

$$= n^2 \int dx\,\frac{1}{\sqrt{1-x^2}}\,T_n(x)\,T_m(x)$$

$$= \int_{-1}^{1} dx\left(-\frac{d}{dx}\sqrt{1-x^2}\,\frac{d}{dx}\,T_n(x)\right)T_m(x)$$

integrate by parts

$$= \int_{-1}^{1} dx\,\sqrt{1-x^2}\left(\frac{d}{dx}T_n(x)\right)\left(\frac{d}{dx}T_m(x)\right)$$

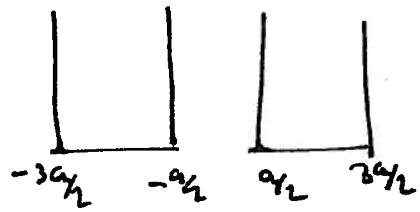
$\sqrt{1-x^2} = 0$ at the boundary

$$= \int_{-1}^{1} dx\quad T_n(x)\left(-\frac{d}{dx}\sqrt{1-x^2}\,\frac{d}{dx}\,T_m(x)\right)$$

$$= m^2 \int dx\,\frac{1}{\sqrt{1-x^2}}\,T_n(x)\,T_m(x)$$

so if $n \neq m$ then $\displaystyle\int_{-1}^{1} dx\,\frac{1}{\sqrt{1-x^2}}\,T_n(x)\,T_m(x) = 0$

3.) a.) For $W \to \infty$ we have two decoupled square well problems



The lowest energy eigenfunction in the left-hand well is

$$\psi_{1L}(x) = \cos\left[\frac{\pi}{a}(x+a)\right] \quad -\frac{3a}{2} < x < -\frac{a}{2}, \ 0 \text{ otherwise}$$

The lowest energy eigenfunction in the right-hand well is

$$\psi_{1R}(x) = \cos\frac{\pi}{a}(x-a) \quad \frac{a}{2} < \frac{3a}{2}, \ 0 \text{ otherwise}$$

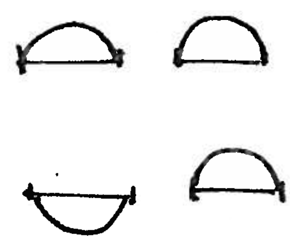
Both have energy

$$E_1 = \frac{\hbar^2\pi^2}{2ma^2}$$

The even and odd eigenstates are

$$\psi_{1\,even} = \frac{1}{\sqrt{2}}\left(\psi_{1L}(x) + \psi_{1R}(x)\right)$$

$$\psi_{1\,odd} = \frac{1}{\sqrt{2}}\left(\psi_{1R}(x) - \psi_{1L}(x)\right)$$



These also have energy E_1

b.) For $a/2 < x < 3a/2$ the Schrödinger equation is

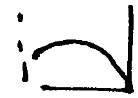
$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$$

or

$$\frac{d^2}{dx^2}\psi = -\frac{2mE}{\hbar^2}\psi$$

a solution that vanishes at $x = 3a/2$ is

$$\psi = B \sin k\left(3a/2 - x\right)$$



with $$k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$$

For $x < a/2$ and $E < W$, the Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + W\psi = E\psi$$

$$\frac{d^2}{dx^2}\psi = + \left[\frac{2m}{\hbar^2}(W-E)\right]\psi$$

the <u>even</u> solution is

$$\psi = A \cosh lx \qquad l = \left[\frac{2m}{\hbar^2}(W-E)\right]^{1/2}$$

c.) At $x = a/2$.

the values of the function must match:

$$A \cosh l\frac{a}{2} = B \sin ka$$

the first derivatives must match:

$$l A \sinh l\frac{a}{2} = -kB \cos ka$$

Dividing:

$$\frac{1}{l}\coth\frac{la}{2} = -\frac{1}{k}\tan ka$$

d.) For $W \gg E$ $\quad l \approx \left(\frac{2mW}{\hbar^2}\right)^{\frac{1}{2}}$ (fixed)

Look for

$$k = \frac{\pi}{a} + \delta k$$

$\tan\pi = 0$ and $\tan(\pi + \delta k a) \cong \delta k a$

Also, if $la \sim \left(\frac{2mW}{\hbar^2}\right)^{\frac{1}{2}} a \gg 1$ then

$$\coth la \approx 1$$

Our equation becomes

$$\frac{1}{l} = -\frac{a}{\pi}\,\delta k\, a$$

or

$$\delta k = -\frac{\pi}{a^2 l} = -\frac{\pi}{a^2}\frac{\hbar}{[2mW]^{\frac{1}{2}}}$$

$$E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2\left(1 - \frac{\hbar}{a[2mW]^{\frac{1}{2}}}\right)^2$$

$$= E_1\left(1 - \frac{2\hbar}{a[2mW]^{\frac{1}{2}}} + \cdots\right)$$

So

$$\delta k = -\frac{\pi}{a^2}\frac{\hbar}{[2mW]^{\frac{1}{2}}}$$

both are negative

$$\delta E = E - E_1 = -\frac{2\hbar}{a[2mW]^{\frac{1}{2}}} E_1$$

e.) For an odd eigenfunction, the solution for $x < a/2$ would be

$$\psi(x) = A \sinh lx$$

Then the equation linking k and l is

$$\frac{1}{l} \tanh \frac{la}{2} = -\frac{1}{k} \tan ka$$

But if $la/2 \gg 1$ $\tanh \frac{la}{2} \approx \coth \frac{la}{2} \approx 1$ and these equations give the same result.

f.) More carefully

$$\tanh \frac{la}{2} \cong 1 - 2e^{-la} + \cdots$$

$$\coth \frac{la}{2} \cong 1 + 2e^{-la} + \cdots$$

so actually

$$(\delta k)_{even} = -\frac{\pi}{a} \cdot \frac{\hbar}{a[2mW]^{1/2}} \left(1 + 2e^{-\left[\frac{2mW}{\hbar^2}\right]^{1/2} a} + \cdots\right)$$

$$(\delta k)_{odd} = -\frac{\pi}{a} \frac{\hbar}{a[2mW]^{1/2}} \left(1 - 2e^{-\left[\frac{2mW}{\hbar^2}\right]^{1/2} a} + \cdots\right)$$

$$(\delta k)_{odd} - (\delta k)_{even} \cong +\frac{\pi}{a} \frac{\hbar}{a[2mW]^{1/2}} \cdot 4e^{-\left[\frac{2mW}{\hbar^2}\right]^{1/2} a}$$

then $$E_{odd} > E_{even}$$ and

$$\Delta E = E_{odd} - E_{even} = E_1 \cdot \frac{2\hbar}{a\,[2mW]^{1/2}} \cdot 4\,e^{-\left[\frac{2mW}{\hbar^2}\right]^{1/2} a}$$

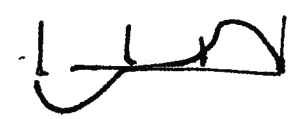
g.) If

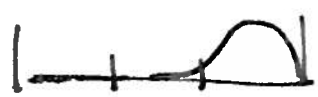
$$\psi_R(x) = \frac{1}{\sqrt{2}}\left(\psi_{1e}(x) + \psi_{1o}(x)\right) \quad \text{at} \quad t=0$$

then at later times this wavefunct is:

$$= \frac{1}{\sqrt{2}}\, e^{-i\frac{E}{\hbar}t}\left(\psi_{1e}(x) + e^{-i\frac{\Delta E}{\hbar}t}\,\psi_{1o}(x)\right)$$

At $t=0$ we have

ψ_{1e}  ψ_{1o} 

$\psi_R(x) \approx$ 

but after a time such that $$e^{-i\frac{\Delta E}{\hbar}t} = -1$$

we have

$\psi_R(x,t) =$ 

h.) The particle has tunneled through the barrier

The time to tunnel is given by

$$e^{-i\frac{\Delta E}{\hbar}t} = -1 \quad \text{or} \quad t = \frac{\hbar}{\Delta E}\pi$$

$$t = \pi\hbar \frac{1}{E_1} \frac{a\,[2mW]^{\frac{1}{2}}}{2\hbar} \frac{1}{4} \exp\left[+ \left[\frac{2mW}{\hbar^2}\right]^{\frac{1}{2}} a \right]$$

$$= \frac{\pi a}{8} \frac{2ma^2}{\hbar^2\pi^2} [2mW]^{\frac{1}{2}} \exp\left[\left[\frac{2mW}{\hbar^2}\right]^{\frac{1}{2}} a \right]$$

$$t = \frac{ma^3}{4\pi\hbar} \left[\frac{2mW}{\hbar^2}\right]^{\frac{1}{2}} \exp\left[\left[\frac{2mW}{\hbar^2}\right]^{\frac{1}{2}} a \right]$$