Physics 130 – Midterm Exam

The exam is worth a total of 100 points.

This exam is closed-book and closed-computer. You may use a one-page crib sheet, which I hope you prepared in advance in accord with the instructions announced on CourseWork.

1. (20 points)

   Sketch the form of the following Hydrogen atom wavefunctions. Indicate clearly the regions in which the wavefunction is positive, negative, and, especially, zero. You might want to include multiple graphs (e.g., a graph of $F(r)$ vs. r), but please always include a sketch of $|\psi(\vec{r})|^2$ in 3 dimensions.

   (a) The 2P, $m = 0$ state.

   (b) The (real-valued) linear combination (3D, $m = 2$) + (3D, $m = -2$)

   (c) The 4S state.

   (d) The 4F, $m = 0$ state.

2. (30 points) The *Chebyshev polynomials* are polynomial solutions of the eigenvalue problem

   $$\left[-(1-x^2)\frac{d^2}{dx^2} + x\frac{d}{dx}\right] f(x) = \lambda f(x)$$

   (a) Show that $T_0(x) = 1$ and $T_1(x) = x$ are solutions of this equation with $\lambda = 0, 1$ respectively.

   (b) Show that the equation has polynomial solutions

   $$T_n(x) = x^n + a x^{n-2} + \dots$$

   with appropriate values of λ, and find $\lambda(n)$ for each positive integer value of n. [Note: outside of this exam, the Chebyshev polynomials are defined with a different normalization convention.]

   (c) Find complete expressions for $T_2(x)$ and $T_3(x)$.

   (d) (not so easy) Show that the Chebyshev polynomials obey the relation

   $$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}\, T_n(x) T_m(x) = 0$$

   for $n \neq m$.

3. (50 points) Consider the Schrödinger equation in one dimension in the double-well potential

   $$V(x) = \begin{cases} W & |x| < a/2 \\ 0 & a/2 < |x| < 3a/2 \\ \infty & 3a/2 < |x| \end{cases}$$

   This potential has two square wells and a barrier of large but finite height.

(a) In the limit $W \to \infty$, the potential has two completely separated square wells of size a. Write explicitly the two lowest-energy eigenstates in this potential. Show that they can be written as even and odd functions of x, as required for a symmetric potential. Both states have energy

$$E_1 = \frac{\hbar^2 \pi^2}{2ma^2}$$

(b) If W is merely large and finite ($W \gg E_1$), the wavefunction need not be zero at $x = a/2$. For an even solution, show that the wavefunction must be of the form

$$\psi(x) = \begin{cases} A \cosh \ell x & 0 < x < a/2 \\ B \sin k(3a/2 - x) & a/2 < x < 3a/2 \end{cases}$$

for appropriate k, ℓ. Write the relation between k and the energy E of the eigenstate, and between ℓ and the energy E.

(c) Write the conditions for matching these two solutions across $x = a/2$. Reduce these to an equation that relates k and ℓ.

(d) Assuming that $W \gg E_1$, k will probably be close to $k_1 = \pi/a$. Solve for k to first order for large W, in the form

$$k = k_1 + \delta k$$

Is δk positive or negative? Find $(E - E_1)$; is this positive or negative?

(e) In the same approximation, find the energy of the lowest odd eigenfunction. Show that the result is that same as that found in (d).

(f) Find the small difference in energy between the lowest even and odd eigenfunctions. You might find useful the expression

$$\tanh z = 1 - 2e^{-2z} + \cdots \quad \text{as} \quad z \to \infty$$

Which eigenfunction has a lower energy eigenvalue?

(g) Let $\psi_{1e}(x)$ and ψ_{1o} be the eigenfunctions for these two cases. Consider the wavefunction

$$\psi_R(x) = \frac{1}{\sqrt{2}}(\psi_{1e}(x) + \psi_{1o}(x))$$

If this is the initial condition for the Schrödinger equation, sketch the wavefunction at $t = 0$ and at several times $t > 0$.

(h) The quantum particle tunnels from one side of the barrier to the other! Estimate the time it takes to tunnel.