General Theory of Angular Momentum

In the previous lecture, I quoted the following theorem: The irreducible finite dimensional unitary representations of the rotation group $SO(3)$ are characterized by a number j which takes integer and half-integer values:

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

The representation labelled by j is called the *spin j representation.* It has $(2j+1)$ states. The eigenvalues of J^z in this representation are

$$J^z/\hbar = -j, -j+1, \ldots \quad j-1, j$$

and, on all states

$$J^2/\hbar^2 = j(j+1)$$

In this lecture, I will give a proof of this theorem. In the process, I will introduce some technology that will be useful for working with angular momentum in quantum mechanics.

Since eigenvalues of J^i are proportional to $\hbar$ and eigenvalues of J^2 are proportional to $\hbar^2$, it will be obvious where all of the $\hbar$'s go in this lecture. Then, to simplify the equations, I will set $\hbar = 1$ in this analysis. We can put the $\hbar$'s back at the end.

To begin, we need to rewrite the angular momentum algebra

$$[J^i, J^j] = i\epsilon^{ijk} J^k$$

in a more convenient form. Let

$$J^+ = J^x + iJ^y \qquad J^- = J^x - iJ^y$$

Then

$$[J^z, J^+] = [J^z, J^x] + i[J^z, J^y]$$
$$= iJ^y + i(-iJ^x) = +J^+$$

So, we find

$$[J^z, J^+] = +J^+ \qquad [J^z, J^-] = -J^-$$

The first of these equations can be written out as

$$J^z J^+ = J^+(J^z + 1)$$

The second is

$$J^z J^- = J^-(J^z - 1)$$

In words, the first equation says that if we measure J^z, add 1, and then act with J^+, we get the same result if we first act with J^+ and then measure J^z. In other words, acting on a J^z eigenstate,

> J^+ raises the eigenvalue of J^z by 1
> J^- lowers the eigenvalue of J^z by 1

Finally, we need the commutation relation of J^+ and J^-. This is

$$[J^+, J^-] = [J^x + iJ^y, J^x - iJ^y]$$

$$= [J^x, -iJ^y] + [iJ^y, J^x] = J^z + J^z = 2J^z$$

The complete algebra is

$$[J^z, J^+] = J^+ \qquad [J^z, J^-] = -J^- \qquad [J^+, J^-] = 2J^z$$

There are three nonzero commutation relations, as in the original form. Notice that J^+ and J^- are not self-adjoint; rather

$$(J^+)^\dagger = J^- \qquad (J^-)^\dagger = J^+$$

Now, I would like to explicitly construct a finite-dimensional representation of this algebra. Once we have a representation of the algebra, we can construct a representation of $SO(3)$, as I explained in the previous lecture. The representation I will construct will manifestly be irreducible, since the set of states cannot be consistently divided into two subspaces.

To begin, diagonalize J^z. I claim that there must be a J^z eigenstate with the property that

$$J^+|j\rangle = 0$$

If there were no such state, then we could pick any eigenstate of J^z and act repeatedly with J^+ as many times as we wished. If $|m\rangle$ were an eigenstate of J^z with eigenvalue m, the state

$$(J^+)^n |m\rangle$$

would be an eigenstate of J^z with eigenvalue $(m+n)$. These states would all be distinct, since they would have different J^z eigenvalues, and there would be an infinite number of them. So, for a finite dimensional representation, there must be a J^z eigenstate that is annihilated by J^z. To continue, I will assume the following properties for this state:

$$J^+|j\rangle = 0 \qquad J^z|j\rangle = j|j\rangle \qquad \langle j|j\rangle = 1$$

The second equation defines j, which at this point is can be an arbitrary real number. The state $|j\rangle$ is called the *highest weight state* of the representation.

Now apply J^- to the state $|j\rangle$. This gives a state with $J^z = (j-1)$. To normalize this state, we need

$$\begin{aligned}
\| J^-|j\rangle\|^2 &= \langle J^- j | J^- j\rangle = \langle j | J^+ J^- | j\rangle \\
&= \langle j | J^+J^- - J^-J^+ + J^-J^+ | j\rangle \\
&= \langle j | [J^+, J^-] + 0 | j\rangle = \langle j | 2J^z | j\rangle = 2j
\end{aligned}$$

Since

$$\| J^-|j\rangle\|^2 \geq 0$$

this proves that either (i) $j = 0$ or (ii) $j > 0$. If $j = 0$, we have a trivial 1-dimensional representation of $SO(3)$ with

$$J^3|j\rangle = 0 \qquad J^+|j\rangle = 0 \qquad J^-|j\rangle = 0$$

This is exactly the *scalar representation* defined in the previous lecture.

If $j > 0$, the state $J^-|j\rangle$ exists. The normalized version of this state is

$$|j-1\rangle = \frac{1}{\sqrt{2j}} J^-|j\rangle$$

We can then apply J^- again. The norm of the resulting state is proportional to

$$\| (J^-)^2|j\rangle\|^2 = \langle j|(J^+)^2(J^-)^2|j\rangle$$

To compute this, move the second factor of J^+ to the right. If it hits $|j\rangle$, the result is zero. Hence

$$\langle j|(J^+)^2(J^-)^2|j\rangle = \langle j|J^+[J^+,J^-]J^- + J^+J^-[J^+,J^-]|j\rangle$$

$$= \langle j|J^+(2J^z)J^- + J^+J^-(2J^z)|j\rangle$$

$$= \langle j|J^+J^-|j\rangle \cdot [2(j-1)+2j] = \langle j|J^+J^-|j\rangle \cdot 4(j-\tfrac{1}{2})$$

Then, either (i) $j = \frac{1}{2}$, or (ii) $j > \frac{1}{2}$. The remaining case $0 < j < \frac{1}{2}$ is eliminated by the positivity of the norm of any state. The case $j = \frac{1}{2}$ gives a 2-dimensional representation of $SO(3)$ in which the two states have J^z eigenvalues

$$J^z = -\tfrac{1}{2},\ +\tfrac{1}{2}$$

This is the *spinor representation* constructed from the air in the previous lecture.

It is clear how this process continues. For the general step, we need to compute

$$\| (J^-)^n |j\rangle \|^2 = \langle j| (J^+)^n (J^-)^n |j\rangle$$

$$= \langle j| (J^+)^{n-1} \{ [J^+,J^-](J^-)^{n-1} + \cdots + (J^-)^{n-1}[J^+,J^-] \} |j\rangle$$

$$= \langle j|(J^+)^{n-1}(J^-)^{n-1}|j\rangle \cdot \{ 2[j-(n-1)] + 2(j-(n-2)) + \cdots + 2(j-1) + 2j \}$$

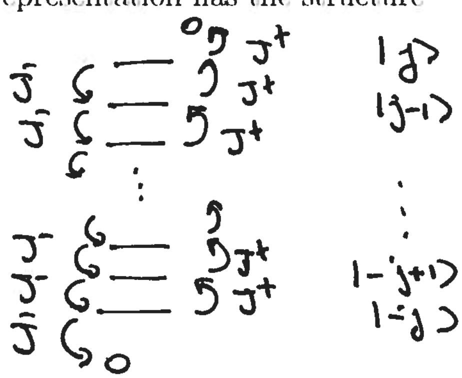
so that

$$= \langle j|(J^+)^{n-1}(J^-)^{n-1}|j\rangle \left(2nj - 2\frac{(n-1)n}{2}\right)$$

$$\|(J^-)^n|j\rangle\|^2 = \|(J^-)^{n-1}|j\rangle\|^2 \cdot 2n\left(j - \frac{n-1}{2}\right)$$

The process must terminate; otherwise we will generate an infinite number of states with negative values of J^z. Then j must be an integer or half-integer.

The complete representation has the structure



The highest-weight state is annihilated by J^+; the lowest-weight state, which has J^z eigenvalue $(-j)$, is annihilated by J^-. Each state below $|j\rangle$ is created by acting J^- on state just above it above. Conversely, we have seen in the algebra above that acting J^+ on a state in this ladder strips off one factor of J^-. Thus, each state above the lowest one is created by acting J^+ on the state just below it.

This is the *spin j representation* of angular momentum or $SO(3)$. It is conventional to label the states of this representation as $\{|jm\rangle\}$. The state $|jm\rangle$ satisfies

$$J^z |jm\rangle = m\,|jm\rangle$$

with m taking the values

$$m = -j,\ -j+1\ \cdots,\ j-1,\ j$$

The algebraic argument above implies that

$$\| J^- |jm\rangle \|^2 = n(2j-n+1)\ \| |jm\rangle \|^2 \qquad n = j-m$$

$$= [(j-m)(j+m-1)]\ \| |jm\rangle \|^2$$

$$= [j(j+1) - m(m-1)]\ \| |jm\rangle \|^2$$

so

$$|j\, m-1\rangle = \frac{1}{[j(j+1) - m(m-1)]^{1/2}}\ J^- |jm\rangle$$

Then, the action of J^- can be written

$$J^- |jm\rangle = [j(j+1) - m(m-1)]^{1/2}\ |j\, m-1\rangle$$

Since J^+ is the adjoint of J^-, its action on a state $|jm\rangle$ is

$$J^+ |jm\rangle = [j(j+1) - m(m+1)]^{1/2}\ |j\, m+1\rangle$$

These formulae give all nonzero matrix elements of J^+, J^-, and J^z and thus present the spin j representation in a completely explicit form.

There is only one task that remains. We need to compute J^2. In terms of our current basis of operators

$$J^2 = (J^x)^2 + (J^y)^2 + (J^z)^2$$

$$= \frac{1}{2}\left[(J^x + iJ^y)(J^x - iJ^y) + (J^x - iJ^y)(J^x + iJ^y)\right] + (J^z)^2$$

or

$$J^2 = \frac{1}{2}\left(J^+J^- + J^-J^+\right) + (J^z)^2$$

Since

$$[J^2, J^-] = 0$$

the value of J^2 is the same on all states of the spin j representation. So, we can evaluate this expression applied to $|jj\rangle$. The three terms give

$$J^2|jj\rangle = \frac{1}{2} J^+J^-|jj\rangle + 0 + j^2|jj\rangle$$

$$= \frac{1}{2}\cdot 2j|jj\rangle + j^2|jj\rangle$$

Thus

$$J^2|jj\rangle = j(j+1)|jj\rangle$$

and so

$$J^2 |jm\rangle = j(j+1) \; |jm\rangle$$

for all m.

We can use this technology to solve the following problem, which arises in many examples in quantum mechanics. If we combine two quantum systems, each of which belongs to a specific irreducible representation of angular moment, what is the total angular momentum of the complete system?

An example of this problem is given by an electron in a state (n, ℓ, m) of the Hydrogen atom. The electron is in a spin ℓ state of orbital angular momentum but also in a spin $\frac{1}{2}$ state of spin angular momentum. We can label the electron states as

$$| \, (\ell, m) \; (\tfrac{1}{2}, m_s) \, \rangle \quad \text{or} \quad | \ell \tfrac{1}{2} \, m \, m_s \rangle$$

There are $2(2\ell + 1)$ states in this Hilbert space. The total angular momentum is

$$J^i \, | \ell \tfrac{1}{2} \, m \, m_s \rangle = L^i_{mm'} \, | \ell \tfrac{1}{2} \, m' m_s \rangle + S^i_{m_s m_s'} \, | \ell \tfrac{1}{2} \, m \, m_s' \rangle$$

We would like to write the $2(2\ell + 1)$ state in terms of their $\vec{J}$ quantum numbers.

The $2(2\ell + 1)$ states that we are discussing here do not form a single irreducible representation of $\vec{J}$. Rather, they can be divided into two separate irreducible representations. We can construct those representations using the tools given in this lecture.

Begin with the state with $m = \ell$, $m_s = +\frac{1}{2}$.

$$| \ell \tfrac{1}{2} \, \ell \, \tfrac{1}{2} \rangle$$

This is the unique state in the Hilbert space with the maximum possible value of J^z, $J^z = \ell + \frac{1}{2}$. Thus, it must be the highest-weight vector of a spin $(\ell + \frac{1}{2})$ representation. We can find the other states of this irreducible representation by acting on this state with

$$J^- = L^- + S^-$$

Here is the construction of the state with $J^z = (\ell - \frac{1}{2})$:

$$\begin{aligned}
J^- |\ell \tfrac{1}{2}\, \ell \tfrac{1}{2}\rangle &= L^- |\ell \tfrac{1}{2}\, \ell \tfrac{1}{2}\rangle + S^- |\ell \tfrac{1}{2}\, \ell \tfrac{1}{2}\rangle \\
&= [\ell(\ell+1) - \ell(\ell-1)]^{\frac{1}{2}} |\ell \tfrac{1}{2} (\ell-1) \tfrac{1}{2}\rangle + [\tfrac{3}{4} - \tfrac{1}{2}(-\tfrac{1}{2})]^{\frac{1}{2}} |\ell \tfrac{1}{2}\, \ell -\tfrac{1}{2}\rangle \\
&= [2\ell]^{\frac{1}{2}} |\ell \tfrac{1}{2}\, \ell-1\, \tfrac{1}{2}\rangle + [1]^{\frac{1}{2}} |\ell \tfrac{1}{2}\, \ell -\tfrac{1}{2}\rangle
\end{aligned}$$

The normalized state is then

$$\frac{1}{\sqrt{2\ell+1}} \left(\sqrt{2\ell}\, |\ell \tfrac{1}{2}\, \ell-1\, \tfrac{1}{2}\rangle + |\ell \tfrac{1}{2}\, \ell -\tfrac{1}{2}\rangle \right)$$

Notice that this state is a linear combination of states with different values of m and m_s.

We have now accounted for one of the two states with $J^z = (\ell - half)$ in the original set of states. The state with the same J^z eigenvalue that is orthogonal to this one is

$$\frac{1}{\sqrt{2\ell+1}} \left(- |\ell \tfrac{1}{2}\, \ell-1\, \tfrac{1}{2}\rangle + \sqrt{2\ell}\, |\ell \tfrac{1}{2}\, \ell -\tfrac{1}{2}\rangle \right)$$

This must be the highest weight vector of another spin representation, one with $j = (\ell - \frac{1}{2})$. You can check this explicitly by showing that J^+ acting on this state gives 0. The lower states of this representation are constructed by applying J^- repeatedly to this state.

In all, we have constructed a spin $(\ell+\frac{1}{2})$ multiplet, with $(2\ell+2)$ states, and a spin $(\ell-\frac{1}{2})$ multiplet, with 2ℓ states. This acccount for all of the states in the original problem. The $j=(\ell+\frac{1}{2})$ states may be thought of as those in which the orbital and spin angular momenta are aligned. The $j=(\ell-\frac{1}{2})$ states may be thought of as those in which the orbital and spin angular momenta are anti-aligned.

We describe a problem of this type as decomposing a *product* of two spin representations into a *sum* of irreducible representations. A notation for the exercise that we have just done is

$$\ell \otimes \frac{1}{2} = (\ell+\tfrac{1}{2}) \oplus (\ell-\tfrac{1}{2})$$

that is, the product of the spin ℓ and spin $\frac{1}{2}$ representations can be representated as a spin $(\ell+\frac{1}{2})$ representation and an spin $(\ell-\frac{1}{2})$ representation. I have explained this problem in the context of an electron in an atom, but the same formulae apply to the abstract mathematical problem, which may appear in other contexts. The product of a spin ℓ with a spin 1 representation can be worked out in a similar way,

$$\ell \otimes 1 = (\ell+1) \oplus \ell \oplus (\ell-1)$$

To check this, add up the number of states in the three final representations

$$(2\ell+1)\cdot 3 = [2(\ell+1)+1] + (2\ell+1) + [2(\ell-1)+1] \quad \checkmark$$

Since the problem of decomposing products of spin representations into sums comes up very often, there is standard notation for the solution of this problem, and sets of tables available where you can look up the results. The overlap of the states of the spin j_1 and spin j_2 representations with states of define total spin j and $J^z=m$ is given by *Clebsch-Gordan coefficients*

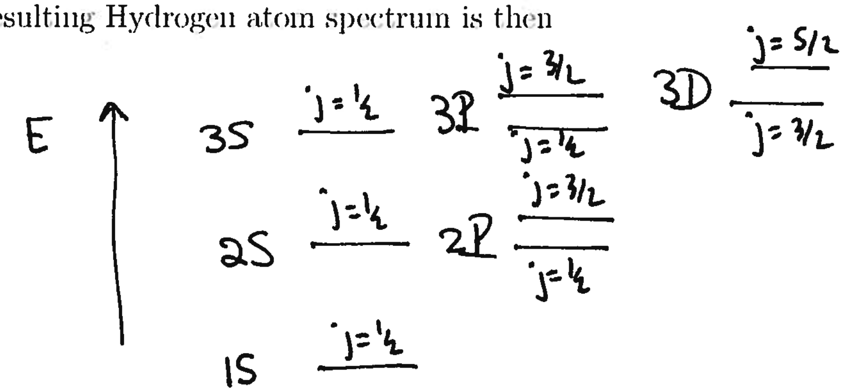
$$\langle j_1 j_2\ m_1 m_2 \,|\, j_1 j_2\ j m\rangle$$

where, always, $m = m_1 + m_2$. For example, the exercise done above, the following Clebsch-Gordon coefficients are computed:

$$\langle \ell\ \tfrac{1}{2}\ \ell\ \tfrac{1}{2} \,|\, \ell\ \tfrac{1}{2}\ (\ell+\tfrac{1}{2})\ (\ell+\tfrac{1}{2})\rangle = 1$$
$$\langle \ell\ \tfrac{1}{2}\ \ell\ {-\tfrac{1}{2}} \,|\, \ell\ \tfrac{1}{2}\ (\ell+\tfrac{1}{2})\ (\ell-\tfrac{1}{2})\rangle = \sqrt{1/2\ell+1}$$
$$\langle \ell\ \tfrac{1}{2}\ \ell-1\ \tfrac{1}{2} \,|\, \ell\ \tfrac{1}{2}\ (\ell+\tfrac{1}{2})\ (\ell-\tfrac{1}{2})\rangle = \sqrt{2\ell/2\ell+1}$$
$$\langle \ell\ \tfrac{1}{2}\ \ell\ {-\tfrac{1}{2}} \,|\, \ell\ \tfrac{1}{2}\ (\ell-\tfrac{1}{2})\ (\ell-\tfrac{1}{2})\rangle = \sqrt{2\ell/2\ell+1}$$
$$\langle \ell\ \tfrac{1}{2}\ \ell-1\ \tfrac{1}{2} \,|\, \ell\ \tfrac{1}{2}\ (\ell-\tfrac{1}{2})\ (\ell-\tfrac{1}{2})\rangle = -\sqrt{1/2\ell+1}$$

A compact table of Clebsch-Gordan coefficients is given in Table 4.8 on p. 188 of Griffiths. The complete explanation of this table, and further results on angular momentum, can be found at: pdg.lbl.gov/2012/reviews/rpp2012-clebsch-gordan-coefs.pdf.

In our study of the Hydrogen atom, we saw that all states with given n have the same energy, of the order of $\alpha^2 mc^2$. As you will learn in Physics 131, corrections to the Hydrogen atom problem from inclusion of special relativity correct this expression to give slightly different energies to states with different total angular moment. The energy splittings are of order $\alpha^4 mc^2$; they are called the Hydrogen atom *fine structure*. The resulting Hydrogen atom spectrum is then



The energy eigenstates are the states of definite total J constructed above. There are further levels of detail in the Hydrogen atom spectrum. These will also be discussed in Physics 131.