Physics 130 - Final Exam

1. (20 points) A spin $\frac{1}{2}$ particle is in the spinor state

$$\chi = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$$

   (a) Show that this state is an eigenstate of $\hat{a} \cdot \vec{S}$, where $\hat{a} = (\sin 60°, 0, \cos 60°) = (\frac{\sqrt{3}}{2}, 0, \frac{1}{2})$ and find the eigenvalue.

   (b) If the spin component S^z is measured, what are the possible outcomes, and with what probability will they appear?

   (c) If the spin component S^x is measured, what are the possible outcomes, and with what probability will they appear?

   (d) If the spin component S^y is measured, what are the possible outcomes, and with what probability will they appear?

   (e) Rotate the spinor χ by 60° about the $\hat{z}$ axis. What is the resulting spinor?

   (f) If the spin components S^z, S^x, S^y are measured for this rotated spinor, what are the possible outcomes, and with what probabilities do they appear?

2. (20 points) In the lectures, we saw that a nonrelativistic Schrödinger electron of mass m in one dimension in a potential

$$V(x) = -\mathbf{V}\delta(x)$$

has a bound state of the form

$$\psi(x) = Ne^{-\kappa|x|}$$

where $\kappa = m\mathbf{V}/\hbar^2$.

   (a) Find the value of N for a normalized wavefunction (in terms of κ).

   (b) We can measure the momentum of the particle in the wavefunction by hitting it suddenly with an energetic photon. It can be shown that a photon of fixed energy can knock out the electron only if the electron has a specific momentum for each specific photon energy. The rate of the knock-out process as a function of the photon energy then measures the probability distribution of p. Compute the probability distribution of the electron momentum p for the wavefunction above.

3. (20 points) Consider a "dimple" potential in 1 dimension with the shape of a harmonic oscillator potential, for small x, and a constant, for large x

$$V(x) = \begin{cases} \frac{1}{2}m\Omega^2 x^2 & |x| < a \\ \frac{1}{2}m\Omega^2 a^2 & |x| > a \end{cases}$$

Let $x_0 = (\hbar/m\Omega)^{1/2}$, and consider $a \gg x_0$. Then, in the region $|x| < a$, the ground state energy $\frac{1}{2}\hbar\Omega$ and the ground state wavefunction will be almost unchanged from its form in the standard harmonic oscillator potential

$$\psi(x) = (\sqrt{\pi}x_0)^{-1/2} \exp[-\frac{1}{2}(x/x_0)^2]$$

(a) With these assumptions, find the differential equation satisfied by the ground state wavefunction for $x > a$.

(b) Find the value of the wavefunction at $x = a$, and, using this data, find the form of the ground state wavefunction for $x > a$.

(c) Imagine that the position x of the particle is measured. Find the probability that the measured position is greater than a.

4. (20 points) The higher-energy states of the Hydrogen atom can make transitions to lower-energy states by emitting photons. The simplest model of these transitions is that the transition *amplitude* is proportional to the matrix element

$$\mathcal{M} = \langle \psi_f | \ \vec{\epsilon} \cdot \vec{r} \, | \psi_i \rangle$$

where $|\psi_i\rangle$, $|\psi_f\rangle$ are the initial and final hydrogen atom states, and $\vec{\epsilon}$ is the polarization vector of the emitted photon. In this problem, you may ignore the electron spin, and you may ignore the direction of motion of the photon, which constrains the direction of $\vec{\epsilon}$. These complications can wait for Physics 131.

The expressions for the Hydrogen atoms 1S, 2S, and 2P wavefunctions are given below after the exam questions.

(a) Compute the matrix element $\mathcal{M}$ for the transitions to the 1S state from the 2S, $2P_x$, $2P_y$, and $2P_z$ states, for photons with $\vec{\epsilon} = \hat{x}$, $\hat{y}$, and $\hat{z}$. There are 12 cases in all, but $\mathcal{M} = 0$ for many of these. As a part of this analysis, show that the rate for the transition $2P_x \to 1S + \gamma(\hat{x})$ is nonzero, where $\gamma(\hat{x})$ is a photon with $\vec{\epsilon} = \hat{x}$.

(b) Give a symmetry argument for one of the zero elements. That is, identify a unitary transformation U that commutes with H, for which $|\psi_f\rangle$ and $\vec{\epsilon} \cdot \vec{r}\,|\psi_i\rangle$ have different quantum numbers.

(c) Put the Hydrogen atom into an electric field in the $\hat{z}$ direction. The electric field couples to the electric dipole moment of the atom and thus adds a term to the Hamiltonian

$$\Delta H = e\vec{E} \cdot \vec{r} = eEz$$

Compute the matrix elements of ΔH between pairs of the four states at $n = 2$. Write this set of matrix elements as a 4×4 matrix and diagonalize it. What are the resulting energies and energy eigenstates?

(d) It can be shown that, if the electric field in (c) is small, its effects on the 1S wavefunction are much smaller than its effect on the 2S and 2P wavefunctions. Assuming this, we can reconsider the computation of the matrix elements $\mathcal{M}$ in

(a) for the wavefunctions $|\psi_i\rangle$ that have been modified by the influence of the external field. Identify one transition for which the rate was previously zero but now is nonzero. Compute the ratio of the rate for this transition to the rate for the transition $2P_x \to 1S + \gamma(\hat{x})$.

5. (20 points) Consider an arrangement of four atoms at the vertices of a tetrahedron. An electron can jump from each atom to any of its neighbors. If the state of an electron sitting on the atom 1 is written as $|1\rangle$, and so forth, a model Hamiltonian for this problem would be given by

$$H\,|a\rangle = -\Delta \sum_{b \neq a} |b\rangle$$

with $\Delta > 0$.

   (a) Write this Hamiltonian as a 4×4 matrix.
   (b) Find the eigenvectors and eigenvalues of this matrix.
   (c) Check that the sum of the eigenvalues is zero. Why is this required?
   (d) Assume that, at $t = 0$, the system is in the state $|4\rangle$. Find the state of the system as a function of time.
   (e) Find the time $t_1 > 0$ at which the probability that the electron is found in the state $|4\rangle$ is again equal to 1.
   (f) Populate the 4 quantum states that you have found in this problem with electrons, assuming for simplicity that electrons do not shift the energy levels of other electrons. The number of distinct orthogonal states with 1 electron is obviously 8, including accounting of the electron spin. How many states are there with 2 electrons? Display these states and find for each the total energy and the total electron spin.

Useful Information

Pauli sigma matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hydrogen atom wavefunctions:

- 1S: $\psi_{1S} = \frac{2}{a_0^{3/2}} \frac{1}{(4\pi)^{1/2}} e^{-r/a_0}$
- 2S : $\psi_{2S} = \frac{1}{\sqrt{2} a_0^{3/2}} \frac{1}{(4\pi)^{1/2}} (1 - \frac{r}{2a_0}) e^{-r/2a_0}$
- 2P$_x$: $\psi_{2P_x} = \frac{1}{\sqrt{24} a_0^{3/2}} \frac{\sqrt{3}}{(4\pi)^{1/2}} \frac{x}{a_0} e^{-r/2a_0}$
- 2P$_y$: $\psi_{2P_y} = \frac{1}{\sqrt{24} a_0^{3/2}} \frac{\sqrt{3}}{(4\pi)^{1/2}} \frac{y}{a_0} e^{-r/2a_0}$
- 2P$_z$: $\psi_{2P_z} = \frac{1}{\sqrt{24} a_0^{3/2}} \frac{\sqrt{3}}{(4\pi)^{1/2}} \frac{z}{a_0} e^{-r/2a_0}$