February 11

What is Quantum Mechanics?

With all of the introduction completed, we are finally ready to discuss the basic principles of quantum mechanics.

In your previous physics courses, your professors told you that a particle is characterized by having a definite position and velocity at any time

$$\vec{x} \qquad \dot{\vec{x}} \qquad \text{at} \quad t = t_0$$

This data gives the initial conditions for a second-order differential equation, Newton's law of motion

$$m\ddot{\vec{x}} = \vec{F}$$

By integrating this equation, we find $\vec{x}(t)$ and $\dot{\vec{x}}(t)$ uniquely at any future time.

Even for field equations, the philosophy is the same. We specify appropriate initial conditions, for example, the values of $\vec{E}(\vec{x})$ and $\vec{B}(\vec{x})$ at a fixed time. Then integrating Maxwell's equations, we find the values of these fields at any future time.

An alternative formulation of mechanics is given by Hamilton's equations. We view the particle as a point in *phase space*

$$(\vec{x}, \vec{p})$$

The evolution of this point is found by integrating the first-order differential equations

$$\dot{\vec{x}} = \frac{\partial H}{\partial \vec{p}} \qquad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{x}}$$

where $H(x,p)$ is the *Hamiltonian function.* For a particle in a potential,

$$\mathcal{H} = \frac{p^2}{2m} + V(x)$$

Then the Hamiltonian equations of motion are

$$\dot{\vec{x}} = \frac{\vec{p}}{m} \qquad \dot{\vec{p}} = -\frac{\partial V}{\partial \vec{x}}$$

These first-order equations are equivalent to Newton's laws for this system.

This philosophy gives a good approximation to many systems, but, fundamentally, it is incorrect. Quantum mechanics gives a very different picture. We still have a differential equation, but the object that is integrated is not a point, and it does not follow a definite trajectory in space. I will now describe the framework that replaces the classical picture.

Here are the basic ideas of quantum mechanics:

1. A quantum mechanical system is described completely as a vector in an appropriate Hilbert space. We call this vector a *quantum state.*

2. Time evolution, and other transformations of the states, are implemented by norm-preserving linear transformations on the vectors, that is, by *unitary transformations.*

3. Measurable properties of the quantum state are represented by self-adjoint (Hermitian) operators that act on the state. An eigenstate of the operator has a definite value of the measurable quantity, equal to the corresponding eigenvalue.

These statements are abstract and subtle. They take a lot of getting used to. The burden of the rest of this course is to help you understand the content of these statements.

In a moment, I will define operators $\mathcal{O}$ representing the position x of a particle, its momentum p, and its energy E. A state will have a definite position, or energy, if it is an eigenstate of the corresponding operator. More generally, the value of an operator $\mathcal{O}$ in a state v, the *expectation value*, is given by

$$\langle \mathcal{O} \rangle_v = \langle v | \mathcal{O} v \rangle$$

The state v can always be written as a linear combination of eigenvectors

$$v = \sum_{n=1} c_n v_n$$

where, if v is normalized,

$$\sum_n |c_n|^2 = 1$$

Then

$$\langle \mathcal{O} \rangle_v = \sum_n \lambda_n |c_n|^2$$

This is always a real number. Our postulates pass the first test: The theory is defined a complex vector space, but output quantities corresponding to measurements are always real-valued. Still, it is not yet clear how to interpret this expectation value. Let me just say for the moment that it represents the average of a large number of measurements with identically prepared states v.

Quantum mechanics uses a special notation for vectors which I will now introduce. The basic quantity evaluated in Hilbert space computations is the inner product

$$\langle v | w \rangle$$

This is a combination of the vectors v and w. Dirac called this a *bracket.* He argued that it is usefully taken apart into two pieces. The first is w, which is vector proper in the Hilbert space, notated

$$|w\rangle$$

called the *ket* vector. The second is v, or $\langle v|\cdot\rangle$, a complex-conjugated vector with inner product. This is notated

$$\langle v|$$

and is called the *bra* vector. The bra is appropriately named. Mathematically, it is a projector in a dual space.

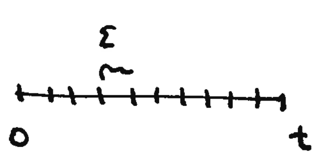
Now I would like to construct the basic operators that act on a quantum state. My argument will be quite abstract, but, as you will see, also general and powerful.

In many system in nature – actually, in all observed systes for which we can ignore the action of external forces – the equations of motion do not depend explicitly on the time. We say that such systems are *time-translation invariant.* Let's investigate the special properties of quantum mechanics in such systems.

In quantum mechanics, time evolution is represented by a unitary transformation. We may write explicitly

$$|v\rangle \rightarrow |v(t)\rangle = U(t)|v\rangle$$

where $U(t)$ is a single operator, independent of $|v\rangle$, acting on the whole Hilbert space. If the equations of motion have no explicit dependence on time, then we can break up the full time interval t into a large number of small time intervals



$$t = N\epsilon$$

and the transformation over each small time interval should be the same,

$$U(t) = \prod_{1}^{N} U(\epsilon)$$

If ϵ is very small, the transformation $U(\epsilon)$ should be close to the identity transformation $U = 1$.

Unitary transformations $U(\epsilon)$ with ϵ so small that terms of order ϵ^2 can be neglected are called *infinitesimal unitary transformations.* Such transformations have a definite structure that I will now describe. We can expand

$$U(\epsilon) = 1 + \epsilon G + \Theta(\epsilon^2)$$

where G is an operator. Since $U(\epsilon)$ is unitary,

$$U^\dagger(\epsilon)\, U(\epsilon) = 1$$

Writing this out

$$(1 + \epsilon G^\dagger + \cdots)(1 + \epsilon G + \cdots) = 1$$

The order ϵ term must vanish, and this implies

$$G + G^\dagger = 0$$

Then any infinitesimal unitary transformation has the form

$$G = -iT$$

$$U(\epsilon) = 1 - i\epsilon T + \cdots$$

where T is a Hermitian operator. The operator T is called the *generator* of the transformation.

We can now build up the finite transformation $U(t)$. Given $U(t)$ for some t, the tranformation for a slightly later time is

$$U(t+\epsilon) = U(\epsilon)\, U(t)$$

Expanding in ϵ, we find

$$U(t) + \epsilon \frac{d}{dt} U(t) + \cdots = (1 - i\epsilon T + \cdots)\, U(t)$$

That is, $U(t)$ satisfies the differential equation

$$\frac{d}{dt} U(t) = -iT\, U(t)$$

The solution of this differential equation can be written in several different ways. First, since T is a Hermitian operator, it has eigenvectors $|v_i\rangle$ and corresponding eigenvalues λ_i. Acting on an eigenvector, the equation becomes

$$\frac{d}{dt} |v_j(t)\rangle = -i\lambda_j\, |v_j(t)\rangle$$

and the solution is

$$|v_j(t)\rangle = e^{-i\lambda_j t}\,|v_j\rangle$$

We can assemble these results to give an operator solution to the differential equation. If V is the unitary transformation that diagonalizes T,

$$T = V^{+}\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} V$$

we define the exponential of T, e^{-iTt} as the operator

$$e^{-iTt} = V^{+}\begin{pmatrix} e^{-i\lambda_1 t} & & & \\ & e^{-i\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{-i\lambda_n t} \end{pmatrix} V$$

Another representation of the solution is given by solving the diferential equation by iteration

$$U(t) = 1 - iTt + \frac{(-i)^2}{2}(Tt)^2 + \frac{(-i)^3}{3!}(Tt)^3 + \ldots$$

We recognize this as the Taylor expansion of the exponential. Actually, this Taylor expansion has an infinite radius of convergence, so it rigorously defines the operator for any value of t. For any unitary transformation built up from infinitesimal transformations, then, we write

$$U(t) = e^{-iTt}$$

and we say that the transformation is *infinitesimally generated*, with T the *generator* of the transformation.

We have now seen that time translation in quantum mechanics has this structure. The generator is conventionally named

$$T\Big|_{\text{time-transl.}} \equiv \frac{H}{\hbar}$$

where the Hermitian operator H is called the *Hamiltonian operator.* It represents the total energy of the system.

A state with definite energy is an eigenstate of H

$$H|v_j\rangle = E_j|v_j\rangle$$

This state will evolve in time according to

$$|v_j(t)\rangle = e^{-it\frac{H}{\hbar}}|v_j\rangle = e^{-it\frac{E_j}{\hbar}}|v_j\rangle$$

A general state $|w\rangle$ in the Hilbert space can be written as a linear combination of eigenstates of H

$$|w\rangle = \sum_j c_j|v_j\rangle$$

The time evolution of this state is

$$|w(t)\rangle = U(t)|w\rangle = \sum_j c_j e^{-i\frac{E_j t}{\hbar}}|v_j\rangle$$

These equations correspond exactly to equations we derived as properties of the solutions of the Schrödinger equation.

Notice that, if $|v\rangle$ is an eigenstate of H at one time, then $|v(t)\rangle$ will be an eigenstate of H at all times. That is, the energy of this state is constant. We have proved that, for any quantum mechanical system for which the equations of motion do not depend on time, energy – defined as the eigenvalue of the Hamiltonian operator – is conserved. This is the analogue of *Noether's theorem* for classical mechanics, that every time-translation-invariant classical system has, by virtue of this property, a conserved energy. This observation solidifies the identification of H with the energy.

An important property of matrix algebra is that, in general, matrix multiplication is not commutative. For example, if A and B are the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then

$$AB = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad BA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

For two matrices, or for two operators on a Hilbert space, we define the *commutator* as

$$[A, B] = AB - BA$$

If $[A, B] = 0$, we say the the matrices or operators *commute*. Every operator commutes with itself

$$[A, A] = 0$$

But the property that A and B commute is a rather special relation.

In fact, let $|v\rangle$ be an eigenvector of a Hermitian operator A with eigenvalue λ. Then if

$$[A, B] = 0$$

but not, in general, otherwise, the state

$$B|v\rangle$$

is an eigenvector of A with the same eigenvalue. Here is the proof :

$$A(B|v\rangle) = (AB - BA)|v\rangle + BA|v\rangle$$

$$= 0 + B \cdot \lambda \cdot |v\rangle$$

$$= \lambda (B|v\rangle)$$

In quantum mechanics, operators A and B that commute are compatible, while operators B that do not commute with A disturb the measured value of A. We will see how this works more specifically in the next lecture.

Energy conservation in quantum mechanics is the relation

$$[H, U(t)] = 0$$

The zero value of the commutator follows from the relation

$$U(t) = e^{-i\frac{Ht}{\hbar}}$$

and the fact that H commutes with itself. The zero commutator implies that, if $|v\rangle$ is an eigenstate of H, then its image at a later time $|v(t)\rangle$ will also be an eigenstate of H.

Other important space-time transformations are space translations and rotations. There are no special places or directions in space, so these should also be generated from infinitesimal transformations. A translation (in one dimension) is parametrized by a distance a. A rotation in three dimensions is parametrized by an axis and an angle of rotation about that axis, that is by a vector $\vec{\theta}$. Translations by a and rotations by $\vec{\theta}$ are implemented by the unitary transformations

$$U(a) \qquad\qquad U(\vec{\theta})$$

Let

$$a = N \cdot \delta \qquad\qquad \vec{\theta} = N \cdot \vec{\xi}$$

Then

$$U(a) = [U(\delta)]^N \qquad\qquad U(\vec{\theta}) = [U(\vec{\xi})]^N$$

Repeating the arguments given above, we can write

$$U(\delta) = 1 - i\delta \frac{P}{\hbar} + \cdots$$

and

$$U(\vec{\theta}) = 1 - i \vec{\xi} \cdot \frac{\vec{J}}{\hbar} + \cdots$$

where P and $\vec{J}$ are new Hermitian operators. I will identify these with the measurements of *momentum* and *angular momentum*, respectively.

This identification implies that P and $\vec{J}$ are naturally conserved in specific circumstances. If a system has no preferred position, it is called *space-translation invariant*. Then, a translated system will have the same time evolution as the original system. An equation that expressed this, for any original state $|v\rangle$, is

$$U(t)\, U(a)\, |v\rangle = U(a)\, U(t)\, |v\rangle$$

This equation states that we get the same result if we translate $|v\rangle$ by a and then time-evolve it or if we time-evolve it and then translate it by a. The infinitesimal version of this equation is

$$\left(1 - i\epsilon \frac{H}{\hbar} + \ldots\right)\left(1 - i\delta \frac{P}{\hbar} + \ldots\right)|v\rangle = \left(1 - i\delta \frac{P}{\hbar} + \ldots\right)\left(1 - i\epsilon \frac{H}{\hbar} + \ldots\right)|v\rangle$$

The term of order $\epsilon\delta$ is

$$HP = PH \quad \text{or} \quad [P, H] = 0$$

Then we find

$$[P, U(t)] = 0$$

that is, a state that is an eigenstate of P at one time remains an eigenstate of P at any later time. Similarly, if a system is *rotationally invariant*, then

$$[\vec{J}, H] = 0$$

This implies the relation

$$[\vec{J}, U(t)] = 0$$

which tells us that a state that is an eigenstate of $\vec{J}$ at any time remains an eigenstate of $\vec{J}$ at any later time.

It is worth making a recap of these results:

1. In a system that is time-translation invariant, there is an operator H that is naturally conserved. We call H the *energy*.
2. In a system that is space-tranlation invariant, there is an operator P that is naturally conserved. We call P the *momentum*.
3. In a system that is rotationally invariant, there is a set of operators $\vec{J}$ that are naturally conserved. We call $\vec{J}$ the *angular momentum*.

These three statements are analogous to the consequences of Noether's Theorem for classical systems.

Angular momentum is somewhat more complicated, and I defer its treatment to a later lecture. In the rest of this lecture, I will study P in one dimension and its relation to the operator X that measures the particle position.

The operator X is a Hermitian operator, and so it has eigenstates $|x_0\rangle$ that represent quantum particles of definite position x_0.

$$X|x_0\rangle = x_0|x_0\rangle$$

We might think of these as Schrödinger wavefunctions that are highly peaked about the position $x = x_0$.

Consider the action on such a state of the translation operator $U(a)$. We expect

$$U(a)|x_0\rangle = |x_0 + a\rangle$$

More generally, on any vector $|v\rangle$, translation by $U(a)$ will shift the value of X by a. An equation that expresses this is

$$X\, U(a)\, |v\rangle = U(a)\, (X + a)\, |v\rangle$$

That is, if we measure X and add a, then translate, that will give the same result as first translating, and then measuring X. The infinitesimal version of this relation is

$$X \left(1 - i\delta \frac{P}{\hbar} + \dots\right) = \left(1 - i\delta \frac{P}{\hbar} + \dots\right)(X + \delta)$$

The term proportional to δ is

$$-\frac{i}{\hbar} XP = -\frac{i}{\hbar} PX + 1$$

or

$$[X, P] = i\hbar$$

This is an important relation called the *canonical commutation relation*. In a moment, we will use this relation to build up the space of state of a particle algebraically.

I will now discuss the time evolution of the expectation value of an operator A in a state $|v\rangle$

$$\langle A \rangle_v (t)$$

We can view this evolution in two different ways. First, we can time-evolve the state $|v\rangle$

$$|v(t)\rangle = e^{-i\frac{Ht}{\hbar}} |v\rangle$$

$$\langle v(t)| = \langle e^{-i\frac{Ht}{\hbar}} v| = \langle v| e^{+i\frac{Ht}{\hbar}}$$

Then the expectation value as a function of time is given by

$$\langle A\rangle(t) = \langle v(t)| A |v(t)\rangle = \langle v| e^{+i\frac{Ht}{\hbar}} A e^{-i\frac{Ht}{\hbar}} |v\rangle$$

This way of computing time evolution is called the *Schrödinger picture*. Time evolution of the expectation value is built from the time evolution of $|v\rangle$. The time evolution of $|v\rangle$, in turn, is computed by integrating the differential equation

$$i\hbar \frac{d}{dt} |v(t)\rangle = H |v(t)\rangle$$

This is called the *Schrödinger equation of motion*. We will see in the next lecture that, for the case of a particle in a potential, it is exactly the Schrödinger equation that we studied earlier in the course.

Equally well, we can compute the expectation value by keeping the state $|v\rangle$ fixed and letting the operator evolve in time

$$\langle A\rangle(t) = \langle v| A(t) |v\rangle$$

according to

$$A(t) = e^{+i\frac{Ht}{\hbar}} A e^{-i\frac{Ht}{\hbar}}$$

This is called the *Heisenberg picture*. The equation of motion for the operator is given by differentiating this equation. Let's compute

$$\frac{d}{dt}A(t) = \left(i\frac{H}{\hbar}\right)\left(e^{i\frac{Ht}{\hbar}}Ae^{-i\frac{Ht}{\hbar}}\right) + \left(e^{i\frac{Ht}{\hbar}}Ae^{-i\frac{Ht}{\hbar}}\right)\left(-i\frac{H}{\hbar}\right)$$

Notice that, while differentiating, I have preserved the ordering of operators. Collecting terms,

$$\frac{d}{dt}A(t) = \left(\frac{-i}{\hbar}\right)\left(A(t)H - H A(t)\right)$$

or

$$i\hbar\frac{d}{dt}A(t) = [A(t), H]$$

This is called the *Heisenberg equation of motion.*

What specific equation of motion does this imply? For concreteness, I will choose the specific form of H motivated by the expression for the energy in particle mechanics

$$H = \frac{P^2}{2m} + V(X)$$

where X and P are now operators.

Next, I will compute the Heisenberg equations of motion for the operators X and P using the canonical commutation relation to evaluate the right-hand side. At time $t = 0$,

$$[X, P] = -i\hbar$$

At any later time, this relation is also true

$$[X(t), P(t)] = \left(e^{i\frac{Ht}{\hbar}} X e^{-i\frac{Ht}{\hbar}}\right)\left(e^{+i\frac{Ht}{\hbar}} P e^{-i\frac{Ht}{\hbar}}\right) - \left(e^{+i\frac{Ht}{\hbar}} P e^{-i\frac{Ht}{\hbar}}\right)\left(e^{+i\frac{Ht}{\hbar}} X e^{-i\frac{Ht}{\hbar}}\right)$$

$$= e^{+i\frac{Ht}{\hbar}} [X, P] e^{-i\frac{Ht}{\hbar}} = e^{+i\frac{Ht}{\hbar}} (i\hbar \mathbb{1}) e^{-i\frac{Ht}{\hbar}} = i\hbar$$

Note that

$$[X, P^2] = X \cdot P \cdot P - PXP + PXP - P \cdot P \cdot X$$

$$= [X, P] \cdot P + P[X, P]$$

$$= i\hbar P + P \cdot i\hbar = 2i\hbar P$$

Similarly

$$[X, P^n] = i\hbar \, n \cdot P^{n-1}$$

and

$$[P, X^n] = -i\hbar \, n X^{n-1}$$

In fact, more generally,

$$[P, f(X)] = -i\hbar \frac{df}{dx}(X)$$

Using these relations, we find

$$i\hbar \frac{d}{dt} X(t) = [X(t), H] = i\hbar \frac{P(t)}{m}$$

and

$$i\hbar \frac{d}{dt} P(t) = [P(t), H] = -i\hbar \frac{dV}{dx}(X)$$

The Heisenberg equations of motion for this system are then

$$\frac{d}{dt} X(t) = \frac{P(t)}{m} \qquad \frac{dP(t)}{dt} = -\frac{dV}{dx}(X(t))$$

These equations imply that, in any state $|v\rangle$,

$$\frac{d}{dt}\langle X\rangle_v(t) = \langle P\rangle_v(t) \qquad \frac{d}{dt}\langle P\rangle_v(t) = -\left\langle \frac{dV}{dx}(X)\right\rangle_v(t)$$

Then if $|v\rangle$ is a macroscopic state in which both X and P take values that are fairly well defined, these quantities

$$\langle X\rangle_v \qquad \langle P\rangle_v$$

obey the classical equations of motion for the position and momentum of a particle.

As usual, the situation simplifies for a harmonic oscillator. The Hamiltonian is

$$H = \frac{P^2}{2m} + \frac{1}{2} m\Omega^2 X^2$$

Then, in any state $|v\rangle$, we have

$$\frac{d}{dt}\langle X\rangle_v = \frac{\langle P\rangle_v}{m} \qquad \frac{d}{dt}\langle P\rangle_v = -m\Omega^2\langle X\rangle_v$$

This is a closed set of equations. When combined, they imply

$$\frac{d^2}{dt^2}\langle X\rangle_v = -\Omega^2\langle X\rangle_v$$

That is, all solutions of the harmonic oscillator problem are periodic

$$\langle X\rangle_v = A\cos(\Omega t + \phi)$$

We saw this from our explicit solution of the Schrödinger equation, but the result is even more obvious here.

Actually, the canonical commutation relation allows us to find the energy spectrum of a harmonic oscillator in a very simple way. First, notice that the eigenvalues of H are all positive. To see this, note that

$$\begin{aligned}\langle v| Hv\rangle &= \langle v|\frac{P^2}{2m}v\rangle + \langle v|\tfrac{1}{2}m\Omega^2X^2|v\rangle \\ &= \frac{1}{2m}\langle v|P^2v\rangle + \tfrac{1}{2}m\Omega^2\langle v|X^2v\rangle \\ &= \frac{1}{2m}\langle Pv|Pv\rangle + \tfrac{1}{2}m\Omega^2\langle Xv|Xv\rangle \\ &= \frac{1}{2m}\|\,|Pv\rangle\|^2 + \tfrac{1}{2}m\Omega^2\,\|\,|Xv\rangle\|^2 > 0\end{aligned}$$

Now define the operators

$$a = \frac{1}{\sqrt{2\hbar m\Omega}}\left(+iP + m\Omega X\right)$$

$$a^{\dagger} = \frac{1}{\sqrt{2\hbar m\Omega}}\left(-iP + m\Omega X\right)$$

These operators satisfy the commutation relation

$$[a, a^{\dagger}] = \frac{1}{2\hbar m\Omega}\left([iP, m\Omega X] + [m\Omega X, -iP]\right)$$

$$= \frac{1}{2\hbar m\Omega}\left(im\Omega\cdot(-i\hbar) + (-im\Omega)\cdot(i\hbar)\right)$$

$$= 1$$

The full algebra of a and $a^{\dagger}$ is

$$[a, a^{\dagger}] = 1 \qquad [a, a] = 0 \qquad [a^{\dagger}, a^{\dagger}] = 0$$

Next, compute

$$\frac{1}{2}\left[a^{\dagger}a + aa^{\dagger}\right] = a^{\dagger}a + \frac{1}{2}$$

Explicitly, this is

$$\frac{1}{2}\frac{1}{2\hbar m\Omega}\left[(-iP + m\Omega X)(iP + m\Omega X) + (iP + m\Omega X)(-iP + m\Omega X)\right]$$

$$= \frac{1}{4\hbar m\Omega}\left[2P^2 + 2m^2\Omega^2 X^2\right]$$

$$= \frac{1}{\hbar\Omega}\left[\frac{1}{2}\frac{P^2}{m} + \frac{1}{2}m\Omega^2 X^2\right]$$

We have now shown that

$$H = \hbar\Omega \cdot (a^\dagger a + \tfrac{1}{2})$$

We can use this representation in the following way: Find a state $|0\rangle$ such that

$$a|0\rangle = 0$$

I will show below that such a state must exist. The state $|0\rangle$ is an eigenstate of H with eigenvalue $E = \frac{1}{2}\hbar\Omega$. Now consider the state

$$a^\dagger|0\rangle$$

I claim that this is an eigenstate of H. This follows from the computation

$$\begin{aligned}
H\, a^\dagger|0\rangle &= \hbar\Omega\, (a^\dagger a + \tfrac{1}{2})\, a^\dagger|0\rangle \\
&= \hbar\Omega\, (a^\dagger a a^\dagger - a^\dagger a^\dagger a + a^\dagger a^\dagger a + \tfrac{1}{2} a^\dagger)|0\rangle \\
&= \hbar\Omega \left[a^\dagger [a, a^\dagger]|0\rangle + a^\dagger a^\dagger a|0\rangle + \tfrac{1}{2} a^\dagger|0\rangle \right] \\
&= \hbar\Omega \cdot (1 + 0 + \tfrac{1}{2})\, a^\dagger|0\rangle
\end{aligned}$$

Indeed, $a^\dagger|0\rangle$ is an eigenstate of H with eigenvalue $E = \frac{3}{2}\hbar\Omega$. By a similar calculation

$$(a^\dagger)^2|0\rangle$$

is an eigenstate of H with eigenvalue $E = \frac{5}{2}\hbar\Omega$, and so forth.

Another way to see this is to compute the commutator of H and $a^\dagger$.

$$[H, a^\dagger] = \hbar\Omega\,[a^\dagger a + \tfrac{1}{2}, a^\dagger]$$

$$= \hbar\Omega\; a^\dagger\,[a, a^\dagger] = \hbar\Omega\, a^\dagger$$

That is,

$$H\,a^\dagger = a^\dagger\,(H + \hbar\Omega)$$

This equation states that, if $|w\rangle$ is an eigenstate of H with energy E_w, then $a^\dagger\,|w\rangle$ is an eigenstate with energy

$$E_w + \hbar\Omega$$

Similarly,

$$[H, a] = -\hbar\Omega a$$

So if $|w\rangle$ is an eigenstate of H with energy E_w, then $a\,|w\rangle$ is an eigenstate with energy

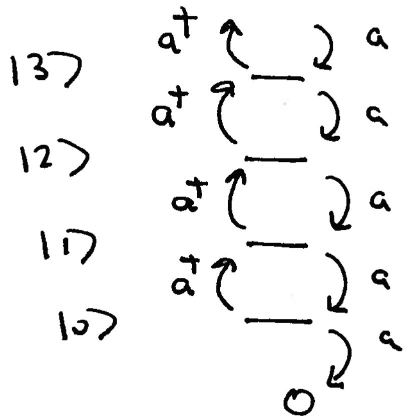
$$E_w - \hbar\Omega$$

We now see that the state $|0\rangle$ defined above must exist. If there were no state $|0\rangle$, we could apply the operator a repeatedly to any energy eigenstate, obtaining energies that would eventually be negative and, later, increasingly negative. This would contradict the result above that all eigenvalues of H are positive.

The eigenstates of the harmonic oscillator Hamiltonian H must therefore consist only of the states

$$(a^{+})^{n}\,|0\rangle$$

built on top of the state $|0\rangle$. These states form a ladder with $|0\rangle$ at the lowest rung



with

$$a|0\rangle = 0 \qquad |n+1\rangle = N(a^{+})^{n}|0\rangle \qquad E_n = (n+\tfrac{1}{2})\hbar\Omega$$

This spectrum of states is just what we found by solving the Schrödinger equation for the harmonic oscillator explicitly.

The formalism that I have developed in the previous few pages also has a romantic history. In the summer of 1925, exhausted from his intense work with Bohr on the phenomenology of atomic spectra, Heisenberg took a seaside vacation in Ostfriesland. There, he had a vision that, if only he would allow X and P to be non-commuting quantities, he could derive the relations that he knew, from his study of spectroscopy, must be correct for the quantum harmonic oscillator. In an intense burst of insight, he wrote the canonical commutation relation and the harmonic oscillator solution just given. (Later authors cleaned up the notation considerably.) Heisenberg wrote it up in a paper. When he returned to Göttingen, he gave this paper to Max Born and asked him to read it, and, if it seemed any good, to publish it. Heisenberg then went off on a previously schedule lecture trip to England—in which these new revelations, too recent and uncertain, were not discussed. Heisenberg did not even recognize that the quantities he was working with were matrices. This idea came later, in a following paper by Born, Jordan, and Kramers.

The canonical commutation relation

$$[X, P] = i\hbar$$

turned out to be the key to the connection between quantum mechanics and classical mechanics. The general form of the connection is simplest to express using the language of the Hamiltonian formulation of mechanics. In that formalism, a mechanical system is described by a point in phase space

$$(q_j, p_j)$$

where $j = 1, \ldots, n$ for n degrees of freedom. The p_j are called *conjugate momenta* for the q_j. These variable obey a formal relation that uses an antisymmetric product called the *Poisson bracket*,

$$\{q_j, p_k\} = \delta_{jk}$$

Through this relation, p_j formally generates translations of q_j. Energy is given by the Hamiltonian function

$$H(q, p)$$

and the equations of motion are written

$$\frac{d}{dt} q_j = \{q_j, H\} \qquad \frac{d}{dt} p_j = \{p_j, H\}$$

For a particle in a potential, the Hamiltonian is that given at the beginning of this lecture, and the Hamiltonian equations evaluate to the equations given there. However, this formalism applies more generally, to any time-translation invariant classical system, even to systems governed by wave equations.

Quantum mechanics makes the formal transformations of Hamilton's theory concrete. The p_j become Hermitian operators that literally generate translations of the q_j. Instead of the Poisson bracket relation above, we have the operator commutation relation

$$[q_j, p_k] = i\hbar\,\delta_{jk}$$

This formula, discovered by Born and Jordan and by Dirac following Heisenberg's work, gives a foundation for the quantum description of any mechanical system.