Problem Set 3 Solution

Laimei Nie, nlm@stanford.edu
Kassahun Betre, kassahun@stanford.edu

February 5, 2013

Phys 130 PS 3 Solution

1\. (a) Let's just skip the algebra here. You can do the integrals by hand, or by Mathematica.

(b) $\langle r \rangle_{n\ell} = \int_0^\infty r^2 dr \cdot |R_{n\ell}(r)|^2 r$

Notice that the r^2 comes from integration in sperical coordinate. The results are:

$\langle r \rangle_{10} = \frac{3}{2}$

$\langle r \rangle_{20} = 6$

$\langle r \rangle_{21} = 5$

$\langle r \rangle_{30} = 27/2$

$\langle r \rangle_{31} = 25/2$

$\langle r \rangle_{32} = 21/2$

why is $\langle r \rangle$ largest at $\ell = 0$ for fixed n ?

Let's look at the function

$$f_{n\ell}(r) = |R_{n\ell}(r)|^2 \cdot r^2$$

Take $n=3$ as an example. You can plot f_{30}, f_{31}, f_{32} using Mathematica and obtain Fig 1.

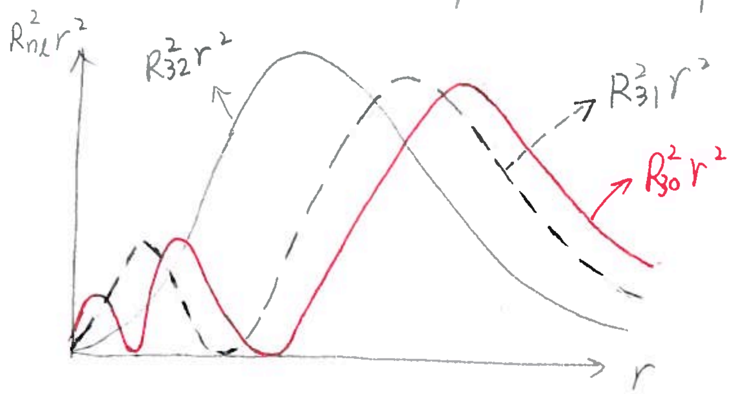


Fig 1. f_{30}, f_{31}, f_{32}.

As we can see, $R_{30}^2 r^2$ has the most extended distribution, so $\int_0^\infty r \cdot R_{30}^2 r^2 dr$ is the largest.

2. $V(x,y,z) = \frac{1}{2} m \Omega^2 (x^2 + y^2 + z^2)$

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(x,y,z) \right] \psi(x,y,z) = E \psi(x,y,z)$$

$$\left[\nabla^2 - \frac{2m}{\hbar^2} V(x,y,z) \right] \psi(x,y,z) = -\frac{2m}{\hbar^2} E \psi(x,y,z) \qquad (1)$$

In rectangular coordinates, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Let $\psi(x,y,z) = X(x) Y(y) Z(z)$,

then,

$$\left[\frac{\partial^2}{\partial x^2} - \frac{m^2 \Omega^2}{\hbar^2} x^2 + \frac{\partial^2}{\partial y^2} - \frac{m^2 \Omega^2}{\hbar^2} y^2 + \frac{\partial^2}{\partial z^2} - \frac{m^2 \Omega^2}{\hbar^2} z^2 \right] X(x) Y(y) Z(z)$$

$$= -\frac{2m}{\hbar^2} E^2 X(x) Y(y) Z(z)$$

After distributing the RHS and dividing the whole equation by $X(x) Y(y) Z(z)$, we get

$$\frac{1}{X(x)}\left(\frac{\partial^2}{\partial x^2} - \frac{m^2\Omega^2}{\hbar^2}x^2\right)X(x) + \frac{1}{Y(y)}\left(\frac{\partial^2}{\partial y^2} - \frac{m^2\Omega^2}{\hbar^2}y^2\right)Y(y)$$

$$+ \frac{1}{Z(z)}\left(\frac{\partial^2}{\partial z^2} - \frac{m^2\Omega^2}{\hbar^2}z^2\right)Z(z) = -\frac{2mE^2}{\hbar^2}$$

$$\Rightarrow \quad \frac{1}{X}\left(\frac{\partial^2}{\partial x^2} - \frac{m^2\Omega^2}{\hbar^2}x^2\right)X = \frac{-2m}{\hbar^2}K_x \qquad (2)$$

$$\frac{1}{Y}\left(\frac{\partial^2}{\partial y^2} - \frac{m^2\Omega^2}{\hbar^2}y^2\right)Y = \frac{-2m}{\hbar^2}K_y \qquad (3)$$

$$\frac{1}{Z}\left(\frac{\partial^2}{\partial z^2} - \frac{m^2\Omega^2}{\hbar^2}y^2\right)Z = \frac{-2m}{\hbar^2}K_z \qquad (4)$$

$K_x + K_y + K_z = E$, where K_1, K_2, K_3 are constant real numbers.

We know the solutions to eqns (1), (2), (3):

$$X_n(x) = \left(\frac{m\Omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(\xi_x)e^{-\xi_x^2/2}, \quad \xi_x = \sqrt{\frac{m\Omega}{\hbar}}\,x$$

$$Y_n(y) = \left(\frac{m\Omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(\xi_y)e^{-\xi_y^2/2}, \quad \xi_y = \sqrt{\frac{m\Omega}{\hbar}}\,y$$

$$Z_n(z) = \left(\frac{m\Omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(\xi_z)e^{-\xi_z^2/2}, \quad \xi_z = \sqrt{\frac{m\Omega}{\hbar}}\,z.$$

$$\Rightarrow \psi_{n_1 n_2 n_3}(x,y,z) = \left(\frac{m\Omega}{\pi\hbar}\right)^{\frac{3}{4}} \frac{1}{\sqrt{2^{(n_1+n_2+n_3)}(n_1!\,n_2!\,n_3!)}}$$

$$\times H_{n_1}(\xi_x)\, H_{n_2}(\xi_y)\, H_{n_3}(\xi_z)$$

$$\times \exp\left\{-\frac{1}{2}\left(\xi_x^2+\xi_y^2+\xi_z^2\right)\right\}$$

Note, that $\xi_x^2+\xi_y^2+\xi_z^2 = \frac{m\Omega}{\hbar}(x^2+y^2+z^2) = \frac{m\Omega}{\hbar} r^2$

So, $$\boxed{\psi_{n_1 n_2 n_3} = N_{n_1 n_2 n_3}\, H_{n_1}(\xi_x)\, H_{n_2}(\xi_y)\, H_{n_3}(\xi_z)\, e^{-\frac{1}{2}\frac{m\Omega}{\hbar} r^2}} \quad (5)$$

Further, $K_x^{n_1} = (n_1+\frac{1}{2})\hbar\Omega$, $K_y^{n_2} = (n_2+\frac{1}{2})\hbar\Omega$, $K_z^{n} = (n+\frac{1}{2})\hbar\Omega$

$$\Rightarrow \boxed{E_{n_1 n_2 n_3} = \left(\frac{3}{2}+n_1+n_2+n_3\right)\hbar\Omega} \quad (6)$$

Ground state is at $n_1 = n_2 = n_3 = 0$

$$\boxed{E_{000} = \frac{3}{2}\hbar\Omega}$$

(b) At the first excited level, there are three states all with energy $\frac{5}{2}\hbar\Omega$. They correspond to the assignments $(n_1, n_2, n_3) = (1,0,0)$ or $(0,1,0)$, or $(0,0,1)$.

i.e. $E_{100} = E_{010} = E_{001} = \frac{5}{2}\hbar\Omega$.

The corresponding states are

$$\psi_{100} = N_{100}\, H_1\left(\sqrt{\frac{m\Omega}{\hbar}}\, x\right) e^{-\frac{1}{2}\frac{m\Omega}{\hbar} r^2}$$

$$\psi_{010} = N_{010}\, H_1\left(\sqrt{\frac{m\Omega}{\hbar}}\, y\right) e^{-\frac{1}{2}\frac{m\Omega}{\hbar} r^2}$$

$$\psi_{001} = N_{001}\, H_1\left(\sqrt{\frac{m\Omega}{\hbar}}\, z\right) e^{-\frac{1}{2}\frac{m\Omega}{\hbar} r^2}$$

$$N_{100} = N_{010} = N_{001} = N_1 = \left(\frac{m\Omega}{\pi\hbar}\right)^{\frac{3}{4}} \frac{1}{\sqrt{2}}$$

$$H_1(\xi) = 2\xi$$

$$\Rightarrow \psi_{100} = 2N_1 \sqrt{\frac{m\Omega}{\hbar}}\; x\; e^{-\frac{1}{2}\frac{m\Omega}{\hbar} r^2}$$

$$\psi_{010} = 2N_1 \sqrt{\frac{m\Omega}{\hbar}}\; y\; e^{-\frac{1}{2}\frac{m\Omega}{\hbar} r^2}$$

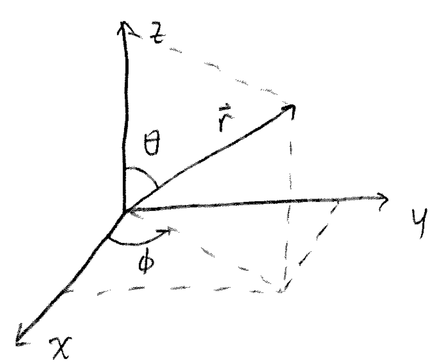
$$\psi_{001} = 2N_1 \sqrt{\frac{m\Omega}{\hbar}}\; z\; e^{-\frac{1}{2}\frac{m\Omega}{\hbar} r^2}$$

In spherical coordinates,

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$



$$x \pm iy = r\sin\theta\,(\cos\phi \pm i\sin\phi)$$

$$= r\sin\theta\, e^{\pm i\phi}, \quad \sin\theta = P_1^1(\cos\theta), \quad P_\ell^m = \text{Associated Legendre}$$

So, $$\boxed{x \pm iy = r\, P_1^1(\cos\theta)\, e^{\pm i\phi} = \mp r\sqrt{\frac{8\pi}{3}}\, Y_1^{\pm 1}} \qquad (7)$$

Also, $z = r\cos\theta$, $P_1^0(\cos\theta) = \cos\theta$

$$\Rightarrow z = r P_1^0(\cos\theta)\, \cancelto{1}{e^{i(0)\phi}} = r\sqrt{\frac{4\pi}{3}}\, Y_1^0(\theta,\phi)$$

$$\boxed{z = \sqrt{\frac{4\pi}{3}}\, r\, Y_1^0} \qquad (8)$$

So, we find that

$$\boxed{F_{11}(r)\, Y_1^{\pm 1}(\theta,\phi) = \left(\mp \psi_{100} - i\,\psi_{010}\right)}$$

$$= 2N_1\sqrt{\frac{m\Omega}{\hbar}}\,(\mp x - iy)\, e^{-\frac{1}{2}\frac{m\Omega}{\hbar}r^2} \qquad (9)$$

and $\boxed{F_{11}(r)\, Y_1^0(\theta,\phi) = \psi_{001}}$

$$= 2N_1\sqrt{\frac{m\Omega}{\hbar}}\, z\, e^{-\frac{1}{2}\frac{m\Omega}{\hbar}r^2} \qquad (10)$$

Combining (8) and (10), we get

$$F_{11}(r)\, Y_1^0(\theta,\phi) = 2N_1\sqrt{\frac{m\Omega}{\hbar}}\, r\, Y_1^0(\theta,\phi)\, e^{-\frac{1}{2}\frac{m\Omega}{\hbar}r^2}$$

$$\Rightarrow F_{11}(r) = 2N_1\sqrt{\frac{m\Omega}{\hbar}}\, r e^{-\frac{1}{2}\frac{m\Omega}{\hbar}r^2}$$

$$\boxed{F_{11}(r) = \sqrt{2}\left(\frac{m\Omega}{\hbar}\right)^{5/2} \pi^{-3/4}\, r\, e^{-\frac{1}{2}\frac{m\Omega}{\hbar}r^2}}$$

(c) From eqn. (6), we have

$$E_{n_1 n_2 n_3} = \left(\frac{3}{2} + n_1 + n_2 + n_3\right)\hbar\Omega$$

$n_1 + n_2 + n_3$ takes the values $0, 1, 2, \cdots$, so

$$E_n = \left(\frac{3}{2} + n\right)\hbar\Omega$$ are the energy levels.

(d) From eqn. (6) the values of n_1, n_2, n_3 that give

$n=2$ are $(n_1, n_2, n_3) = (2,0,0), (0,2,0), (0,0,2),$
$(1,1,0), (1,0,1), (0,1,1).$

So there are 6 states with energy $E = \left(\frac{3}{2}+2\right)\hbar\Omega$.

The states are $\psi_{200}, \psi_{020}, \psi_{002}, \psi_{110}, \psi_{101}, \psi_{011}$.

From eqn. (6), we have,

$$x + iy = -\sqrt{\frac{8\pi}{3}}\, r Y_1^1$$
$$x - iy = \sqrt{\frac{8\pi}{3}}\, r Y_1^{-1}$$

$$\Rightarrow \left.\begin{aligned} 2x &= \sqrt{\frac{8\pi}{3}}\, r \left(Y_1^{-1} - Y_1^1\right) \\ 2iy &= -\sqrt{\frac{8\pi}{3}}\, r \left(Y_1^{-1} + Y_1^1\right) \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x &= \sqrt{\frac{2\pi}{3}}\, r\left(Y_1^{-1} - Y_1^1\right) \\ y &= i\sqrt{\frac{2\pi}{3}}\, r\left(Y_1^{-1} + Y_1^1\right) \\ z &= \sqrt{\frac{4\pi}{3}}\, r\, Y_1^0 \end{aligned}\right\} \quad (11)$$

From (7),

To proceed, let's express the following in terms of spherical harmonics:

1) $x^2 = \frac{2\pi}{3} r^2 \left(Y_1^{-1} - Y_1^{1}\right)^2$,

$$\left(Y_1^{-1} - Y_1^{1}\right)^2 = \left(Y_1^{-1}\right)^2 - 2Y_1^{-1}Y_1^{1} + \left(Y_1^{1}\right)^2,$$

$$\left(Y_1^{\pm1}\right)^2 = \frac{3}{8\pi}\sin^2\theta\, e^{\pm 2i\phi} = \frac{3}{8\pi}\sqrt{\frac{32\pi}{15}}\, Y_2^{\pm2} = \sqrt{\frac{3}{10\pi}}\, Y_2^{\pm2},$$

$$Y_1^{-1}Y_1^{1} = \frac{-3}{8\pi}\sin^2\theta, \quad Y_2^0 = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right) = \sqrt{\frac{5}{16\pi}}\left(2 - 3\sin^2\theta\right)$$

$$\Rightarrow \sqrt{\frac{16\pi}{5}}\, Y_2^0 = 2 - 3\sin^2\theta$$

$$\sqrt{\frac{4}{5\pi}}\, Y_2^0 = \frac{1}{4\pi} - \frac{3}{8\pi}\sin^2\theta$$

$$\Rightarrow Y_1^{-1}Y_1^{1} = \frac{-3}{8\pi}\sin^2\theta = \sqrt{\frac{4}{5\pi}}\, Y_2^0 - \sqrt{\frac{1}{4\pi}}\, Y_0^0$$

So, $x^2 = \frac{2\pi}{3} r^2 \left\{ \sqrt{\frac{3}{10\pi}}\, Y_2^{-2} - 2\sqrt{\frac{4}{5\pi}}\, Y_2^0 + 2\sqrt{\frac{1}{4\pi}}\, Y_0^0 + \sqrt{\frac{3}{10\pi}}\, Y_2^{2} \right\}$

$$= \frac{2}{3}\sqrt{\pi}\, r^2 \left\{ \sqrt{\frac{3}{10}}\left(Y_2^{-2} + Y_2^{2}\right) - 2\left(\sqrt{\frac{4}{5}}\, Y_2^0 + \sqrt{\frac{1}{4}}\, Y_0^0\right) \right\}$$

2) $y^2 = -\frac{2\pi}{3} r^2 \left[(Y_1^{-1})^2 + 2Y_1^{-1}Y_1^1 + (Y_1^1)^2 \right]$

$= -\frac{2\sqrt{\pi}}{3} r^2 \left[\sqrt{\frac{3}{10}} (Y_2^{-2} + Y_2^2) + 2\left(\sqrt{\frac{4}{5}} Y_2^0 + \sqrt{\frac{1}{4}} Y_0^0 \right) \right]$ //

3) $z^2 = \frac{4\pi}{3} r^2 (Y_1^0)^2$

$(Y_1^0)^2 = \frac{3}{4\pi} \cos^2\theta$,

$Y_2^0 = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$

$\Rightarrow \sqrt{\frac{1}{5\pi}} Y_2^0 = \frac{3}{4\pi}\cos^2\theta - \frac{1}{4\pi}$

$(Y_1^0)^2 = \frac{3}{4\pi}\cos^2\theta = \sqrt{\frac{1}{5\pi}} Y_2^0 + \sqrt{\frac{1}{4\pi}} Y_0^0$

$\therefore z^2 = \frac{4\pi}{3} r^2 \left(\sqrt{\frac{1}{5\pi}} Y_2^0 + \sqrt{\frac{1}{4\pi}} Y_0^0 \right)$

$= \frac{4}{3}\sqrt{\pi} r^2 \left(\sqrt{\frac{1}{5}} Y_2^0 + \sqrt{\frac{1}{4}} Y_0^0 \right)$ //

4) $xy = i\frac{2\pi}{3} r^2 \left[(Y_1^{-1})^2 - (Y_1^1)^2 \right]$

$= i\frac{2\pi}{3} r^2 \sqrt{\frac{3}{10\pi}} \left[Y_2^{-2} - Y_2^2 \right]$

$= i\sqrt{\frac{2\pi}{15}} r^2 \left[Y_2^{-2} - Y_2^2 \right]$ //

5) $xz = \frac{2\sqrt{2}\pi}{3} r^2 \left[Y_1^{-1} Y_1^0 - Y_1^1 Y_1^0 \right]$,

$$Y_1^{\pm 1} Y_1^0 = \mp \frac{3}{4\sqrt{2}\pi} \cos\theta \sin\theta \, e^{\pm i\phi}$$

$$= \frac{3}{4\sqrt{2}\pi} \sqrt{\frac{8\pi}{15}} \, Y_2^{\pm 1}$$

$$= \sqrt{\frac{3}{20\pi}} \, Y_2^{\pm 1}$$

$$xz = \frac{2\sqrt{2}\pi}{3} r^2 \sqrt{\frac{3}{20\pi}} \left[Y_2^{-1} - Y_2^1 \right]$$

$$= \sqrt{\frac{2\pi}{15}} \, r^2 \left[Y_2^{-1} - Y_2^1 \right] //$$

6) $yz = \frac{i\, 2\sqrt{2}\pi}{3} r^2 \left[Y_1^{-1} Y_1^0 + Y_1^1 Y_1^0 \right]$

$$= i \sqrt{\frac{2\pi}{15}} \, r^2 \left[Y_2^{-1} + Y_2^1 \right] //$$

Let's now go back to the wave functions.

$$\psi_{200} \sim H_2(\xi_x) = 4\xi_x^2 - 2 = 4\frac{m\Omega}{\hbar} x^2 - 2$$

$$= \frac{4m\Omega}{\hbar} \left[x^2 - \frac{\hbar}{m\Omega} \frac{1}{2} \right] = \frac{4m\Omega}{\hbar} \left[x^2 - \frac{\hbar}{m\Omega} \sqrt{\pi} \, Y_0^0 \right]$$

$$\psi_{020} \sim H_2(\xi_y) = 4\xi_y^2 - 2 = \frac{4m\Omega}{\hbar} y^2 - 2$$

$$= \frac{4m\Omega}{\hbar} \left[y^2 - \frac{\hbar}{m\Omega} \sqrt{\pi} \, Y_0^0 \right]$$

$$\psi_{002} \sim H_2(\xi_z) = \frac{4m\Omega}{\hbar}\left[z^2 - \frac{\hbar}{m\Omega}\sqrt{\pi}\, Y_0^0\right]$$

Including the other $n=2$ wave functions, we have

$$\begin{pmatrix} \psi_{200} \\ \psi_{020} \\ \psi_{002} \\ \psi_{110} \\ \psi_{101} \\ \psi_{011} \end{pmatrix} = \frac{4m\Omega}{\hbar}\sqrt{\frac{\pi}{3}}\, r^2 e^{-\frac{m\Omega}{2\hbar}r^2} \begin{pmatrix} \sqrt{\frac{2}{5}} & 0 & -\frac{8}{\sqrt{15}} & 0 & \sqrt{\frac{2}{5}} & \left(\frac{-2}{\sqrt{3}} + \frac{\hbar}{m\Omega}\frac{\sqrt{3}}{r^2}\right) \\ -\sqrt{\frac{2}{5}} & 0 & -\frac{8}{\sqrt{15}} & 0 & -\sqrt{\frac{2}{5}} & \left(\frac{-2}{\sqrt{3}} + \frac{\hbar}{m\Omega}\frac{\sqrt{3}}{r^2}\right) \\ 0 & 0 & \frac{4}{\sqrt{15}} & 0 & 0 & -\left(-\frac{2}{\sqrt{3}} + \frac{\hbar}{m\Omega}\frac{\sqrt{3}}{r^2}\right) \\ -i\sqrt{\frac{2}{5}} & 0 & 0 & 0 & i\sqrt{\frac{2}{5}} & 0 \\ 0 & -\sqrt{\frac{2}{5}} & 0 & \sqrt{\frac{2}{5}} & 0 & 0 \\ 0 & i\sqrt{\frac{2}{5}} & 0 & i\sqrt{\frac{2}{5}} & 0 & 0 \end{pmatrix} \begin{pmatrix} Y_2^2 \\ Y_2^1 \\ Y_2^0 \\ Y_2^{-1} \\ Y_2^{-2} \\ Y_0^0 \end{pmatrix}$$

We can easily invert the matrix using mathematica to get the Y_ℓ^m's in terms of the ψ's.

(e) For $n=3$, there are 10 states,

$(3,0,0)$, $(0,3,0)$, $(0,0,3)$, $(2,1,0)$, $(2,0,1)$, $(0,2,1)$

$(0,1,2)$, $(1,0,2)$, $(1,2,0)$, $(1,1,1)$

Based on the result in (d), we expect these states to be linear combinations of $F_{n\ell}(r)\, Y_\ell^m$'s. There must be $\ell=3$ states coming from $x^3 \sim \sin^3\theta\, e^{3i\phi}$. There are 7 $\ell=3$ states. The remaining 3 states must be $\ell=1$ states.

(f) For a general n, the number of states is the number of ways of writing n as a sum of 3 numbers, including zeros.

The number of ways of writing n as a sum of k integers, all greater than 0 is given by $\binom{n-1}{k-1}$

In our case, the total number will be the sum of the product of the number of ways of picking a slot to fill with 0 and the number of ways of partitioning n into the remaining slots.

$$\text{No. of states} = \binom{3}{0}\binom{n-1}{3-1} + \binom{3}{1}\binom{n-1}{2-1} + \binom{3}{2}\binom{n-1}{1-1}$$

$$= \binom{n-1}{2} + 3\binom{n-1}{1} + 3\binom{n}{0}$$

$$= \binom{n-1}{2} + 3(n-1) + 3$$

$$= \binom{n-1}{2} + 3n$$

$$= \frac{(n-1)(n-2)}{2} + 3n$$

$$= \frac{n^2+3n+2}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

//

(9) One of the eigenstates is ψ_{n00}

$\psi_{n00} \sim H_n(\xi_x)$ which contains x^n in the expansion.

$x^n \sim (Y_1^{-1} - Y_1^1)^n \supset (Y_1^1)^n \sim \sin^n\theta\, e^{in\phi} = (\text{const})\, Y_n^n$

Therefore, there must be at least one state with $l=n$ in the linear combination.

$$\text{Further, } \psi_{n_1 n_2 n_3}(r) = \overset{\swarrow \text{ Normalization}}{N}\, H_{n_1}(x)\, H_{n_2}(y)\, H_{n_3}(z)\, e^{-\frac{c}{2}r^2} \qquad c = \frac{m\Omega}{\hbar}$$

let $n = n_1 + n_2 + n_3$.

Under $\vec{r} \to -\vec{r}$, i.e., $(x, y, z) \to (-x, -y, -z)$,

$$\psi_{n_1 n_2 n_3}(\vec{r}) \to \psi_{n_1 n_2 n_3}(-\vec{r}) = N H_{n_1}(-x) H_{n_2}(-y) H_{n_3}(-z)\, e^{-\frac{c}{2}r^2}$$

Recall that $H_a(-\xi) = (-1)^a H_a(\xi)$.

$$\text{So, } \psi_{n_1 n_2 n_3}(-\vec{r}) = N (-1)^{n_1+n_2+n_3} H_{n_1}(x) H_{n_2}(y) H_{n_3}(z)\, e^{-\frac{c}{2}r^2}$$

$$= (-1)^n \psi_{n_1 n_2 n_3}(\vec{r})$$

$$\psi_{n_1 n_2 n_3}(\vec{r}) = \sum_l F_{nl}(r)\, Y_l^m(\theta, \phi). \qquad (*)$$

Under $\vec{r} \to -\vec{r}$, $\quad (r, \theta, \phi) \to (r, \theta+\pi, \phi+\pi)$

$$\Rightarrow \cos\theta \to \cos(\theta+\pi) = -\cos\theta$$

$$Y_\ell^m(\theta, \phi) = N_{\ell m} P_\ell^m(\cos\theta)\, e^{im\phi},$$

$$P_\ell^m(x) = (1-x^2)^{|m|/2} \left(\frac{d}{dx}\right)^{|m|} P_\ell(x),$$

$$P_\ell^m(-x) = (1-x^2)^{|m|/2} \left(\frac{d}{dx}\right)^{|m|} (-1)^m P_\ell(-x),$$

$$P_\ell(x) = N_\ell \left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$$

$$\Rightarrow P_\ell(-x) = N_\ell \left(\frac{d}{dx}\right)^\ell (-1)^\ell (x^2-1)^\ell$$

$$\Rightarrow P_\ell^m(-x) = (-1)^m (-1)^\ell (1-x^2)^{|m|/2} \left(\frac{d}{dx}\right)^{|m|} P_\ell(x)$$

$$= (-1)^{m+\ell} P_\ell^m(x)$$

$$\Rightarrow Y_\ell^m(\theta+\pi, \phi+\pi) = N_{\ell m} P_\ell^m(-\cos\theta)\, e^{im(\phi+\pi)}$$

$$= (-1)^{m+\ell} N_{\ell m} P_\ell^m(\cos\theta)\, e^{im\phi}\, \underset{\uparrow \atop (-1)^m}{e^{im\pi}}$$

$$= (-1)^{m+\ell+m}\, Y_\ell^m(\theta, \phi)$$

$$= (-1)^\ell\, Y_\ell^m(\theta, \phi)$$

Let's perform $\vec{r} \to -\vec{r}$ in $(*)$,

$$\Psi_{n_1 n_2 n_3}(-\vec{r}) = \sum_{l} F_{nl}(r)\, Y_l^m(\theta+\pi, \phi+\pi)$$

$$(-1)^n\, \Psi_{n_1 n_2 n_3}(\vec{r}) = \sum_{l} (-1)^l F_{nl}(r)\, Y_l^m(\theta, \phi) \qquad (\checkmark)$$

If n is even, all the l's appearing on the RHS of $(\checkmark)$ better be even. Otherwise $(\checkmark)$ will be a different equation than $(-1)^n (*)$.

If n is odd all the l's on the RHS have to be odd by the same logic. $\boxtimes$

h) Hydrogen

$n \backslash l$	0	1	2	3
1	✓			
2	✓	✓		
3	✓	✓	✓	
4	✓	✓	✓	✓

HMO

$n \backslash l$	0	1	2	3
0	✓			
1	–	✓		
2	✓	–	✓	
3	–	✓	–	✓
4	✓	–	✓	–

In both Hydrogen and HMO, energy depends only on n quantum number. However, some l's are not allowed depending on the n.

3. (a) $r_p = Z^{1/3} \cdot 1.6 \times 10^{-15}\, m = 1.6\times10^{-15}\, m$ for hydrogen.

Then

$n=1, l=0$: $\int_0^{r_p} |R_{10}(r)|^2 r^2 dr$

$$= \int_0^{r_p} 4a^{-3} e^{-2r/a} r^2 dr$$

$$= 4a^{-3}a^3 \int_0^{r_p} e^{-2r/a} \left(\frac{r}{a}\right)^2 d\left(\frac{r}{a}\right)$$

$$= 4 \int_0^{r_p/a} e^{-2x} x^2 dx$$

Since $r_p/a \sim 10^{-5}$, we can expand e^{-2x}:

$e^{-2x} \approx 1-2x$

Then $\int_0^{r_p} |R_{10}(r)|^2 r^2 dr$

$$\approx 4\int_0^{r_p/a} (1-2x)x^2 dx$$

$\approx 3.7\times10^{-14}$ (result may vary if you make

Similarly we have

$P_{20} \equiv \int_0^{r_p} |R_{20}(r)|^2 r^2 dr \sim 10^{-15}$

$P_{21} \sim 10^{-15}$

$P_{30} \sim 10^{-16}$

$P_{31} \sim 10^{-27}$

$P_{32} \sim 10^{-38}$

(b) The probability increases as Z increases, since we have a larger integration range.
Physically, larger Z means more positive charges in the nucleus, which pull the electron closer to the nucleus.

Phys 130 PS 3 Solution Problem 3 (continued)

(c). Bohr radius $a = \dfrac{4\pi\varepsilon_0 \hbar}{\underline{\underline{z}} m e^2}$ Don't forget!

For iron and lead,

$a_{Fe} = \dfrac{a_H}{26}$, $a_{Pb} = \dfrac{a_H}{82}$,

$r_{p,Fe} = (26)^{1/3} r_{p,H}$ $r_{p,Pb} = (82)^{1/3} r_{p,H}$

The probabilities are

iron: $P_{10,Fe} \approx 1.7 \times 10^{-8}$

lead: $P_{10,Pb} \approx 1.6 \times 10^{-6}$

(d) $a_\mu = \dfrac{a_H}{207 \cdot Z}$, $r_{p,\mu} = r_{p,H}$

The probabilities are

hydrogen: $P_{10,H} \approx 3.2 \times 10^{-7}$

iron : $P_{10,Fe} \approx 0.074$

lead : $P_{10,Pb} \approx 0.82$

Note : $P_{10,Fe} \neq (207 \times 26)^3 \cdot P_{10,H}$,

since $P_{10} \not\propto a^{-3}$

$\downarrow$

P_{10} is a dimensionless quantity, so we can not say it is proportional to a^{-3} (the proportionality coefficient must contain a as well).

In fact, $P_{10} = 4a^{-3} \int_0^{r_p} r^2 dr \cdot e^{-r/2a}$

$$= 4a^{-3} a^3 \int_0^{r_p} \left(\frac{r}{a}\right)^2 d\left(\frac{r}{a}\right) \cdot e^{-\frac{r}{2a}}$$

$$= 4 \int_0^{r_p/a} x^2 \, dx \, e^{-\frac{x}{2}} \not\propto a^{-3}$$