Problem Set 2 Solution

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Phys 130 PS 2 Solution

Problem 1

(a) It's easy to show that

$$a_2 = -2, \quad a_3 = -12, \quad a_4 = -48, \quad b_4 = 12$$

(b)
$$S(\xi, s) = e^{-\xi^2} e^{-(\xi-s)^2} = e^{-s^2+2\xi s}$$

$$\frac{\partial}{\partial \xi} S(\xi, s) = e^{-s^2+2\xi s} \cdot 2s$$

$$\frac{\partial^2}{\partial \xi^2} S(\xi, s) = e^{-s^2+2\xi s} \cdot 4s^2$$

$$\frac{\partial}{\partial s} S(\xi, s) = e^{-s^2+2\xi s} \cdot (-2s+2\xi)$$

$$\Rightarrow \frac{\partial^2}{\partial \xi^2} S(\xi, s) - 2\xi \frac{\partial}{\partial \xi} S(\xi, s) + 2s \frac{\partial}{\partial \xi} S(\xi, s)$$

$$= e^{-s^2+2\xi s} \left(4s^2 - 2\xi \cdot 2s + 2s(-2s+2\xi) \right)$$

$$= 0.$$

(c)
$$S(\xi, s) = e^{\xi^2} e^{-(\xi-s)^2} = e^{-\xi^2+2\xi s} = \sum_{n=0}^{\infty} \frac{s^n (2\xi - s)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{s^n}{\cancel{n!}} \sum_m (-s)^m (2\xi)^{n-m} \cdot \frac{\cancel{n!}}{(n-m)!\, m!}$$

Let $k = m+n$, then $\sum_n \sum_m \to \sum_k \sum_m$,

and $$S(\xi, s) = \sum_k \sum_m \frac{s^k}{k!} (-1)^m \cdot \frac{k!}{(k-2m)!\, m!} (2\xi)^{k-2m}$$

$$\Rightarrow \sum_m (-1)^m \frac{k!}{(k-2m)!\, m!} (2\xi)^{k-2m} = H_k(\xi)$$

$\Rightarrow$ $H_k(\xi)$ is a polynomial, and when $m=0$, the term is $2^k \xi^k$.

Now let's prove (3). Plugging $S(\xi,s)=\sum_{n=0}^{\infty}\frac{s^n}{n!}H_n(\xi)$ into

$$\frac{\partial^2}{\partial\xi^2}S(\xi,s)-2\xi\frac{\partial}{\partial\xi}S(\xi,s)+2s\frac{\partial}{\partial s}S(\xi,s)=0,$$

we have

$$\sum_{n=0}^{\infty}\frac{s^n}{n!}\left[\frac{d^2H_n(\xi)}{d\xi^2}-2\xi\frac{dH_n(\xi)}{d\xi}+2nH_n\right]=0$$

thus $H_n(\xi)$ satisfies the Hermite equation with eigenvalue n.

Previously we had $H_n(\xi)=\sum_m(-1)^m\frac{n!}{(n-2m)!\,m!}(2\xi)^{n-2m}$.

Now let's plug in $n=2, 3, 4$

$n=2$: $H_2(\xi)=4\xi^2-2$

$H_3(\xi)=8\xi^3-12\xi$

$H_4(\xi)=16\xi^4-48\xi^2+12$.

Another approach is Taylor expanding $S(\xi,s)$ at $S(\xi,0)$.

$$S(\xi,s)=\sum_n\frac{s^n}{n!}\left.\frac{\partial^nS(\xi,s)}{\partial s^n}\right|_{s=0}$$

Thus
$$H_n(\xi)=\left.\frac{\partial^nS(\xi,s)}{\partial s^n}\right|_{s=0}$$
$$=e^{\xi^2}\left.\frac{\partial^n}{\partial s^n}e^{-(\xi-s)^2}\right|_{s=0}$$
$$=e^{\xi^2}(-1)^n\left.\frac{\partial^n}{\partial\xi^n}e^{-(\xi-s)^2}\right|_{s=0}$$
$$=e^{\xi^2}(-1)^n\frac{d^n}{d\xi^n}e^{-\xi^2}$$

Then it's easy to prove that $H_n(\xi)=2^n\xi^n+\ldots\ldots$,

$H_2(\xi)=4\xi^2-2$, etc.

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Problem 1 (continued)

(d) $$\int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2} S(\xi,s)\, S(\xi,t)$$

$$= \int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2+(2s+2t)\xi} \cdot e^{-(s^2+t^2)}$$

$$= \sqrt{\pi}\, e^{(s+t)^2} \cdot e^{-(s^2+t^2)}$$

$$= \sqrt{\pi}\, e^{2st}$$

$$= \sqrt{\pi} \sum_n \frac{2^n s^n t^n}{n!}$$

On the other hand,

$$\int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2} S(\xi,s)\, S(\xi,t)$$

$$= \int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2} \sum_m \frac{s^m}{m!} H_m(\xi) \cdot \sum_k \frac{t^k}{k!} H_k(\xi)$$

$$= \sum_m \sum_k \frac{s^m t^k}{m!\,k!} \int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2} H_m(\xi) H_k(\xi)$$

Thus

$$\sqrt{\pi} \sum_n \frac{2^n s^n t^n}{n!} = \sum_m \sum_k \frac{s^m t^k}{m!\,k!} \cdot \int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2} H_m(\xi) H_k(\xi)$$

$$\Rightarrow \int_{-\infty}^{+\infty} d\xi\, e^{-\xi^2} H_m(\xi) H_k(\xi) = \delta_{mk} \cdot 2^m m! \cdot \sqrt{\pi}$$

Normalized wavefunction of harmonic oscillator:

$$\psi_n(x) = \underbrace{\left(\frac{m\Omega}{\hbar}\right)^{1/4}} \cdot \left(\frac{1}{2^n n! \sqrt{\pi}}\right)^{1/2} \cdot H_n(\xi)\, e^{-\xi^2/2}, \quad \xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

By dimensional analysis we know this factor must be there.

Because $\int_{-\infty}^{+\infty} |\psi_n(x)|^2 dx = 1 \Rightarrow [\psi_n(x)] = (\text{length})^{-\frac{1}{2}}$

and ξ, $H_n(\xi)$, $e^{-\xi^2/2}$ are all dimensionless,

then we need a constant with dimension $(\text{length})^{-\frac{1}{2}} \to \sqrt[4]{\frac{m\omega}{\hbar}}$.

You can also get it by explicit calculation.

Problem 2

(a) $$\left[z^2 \frac{d^2}{dz^2} + z\frac{d}{dz} + (z^2 - m^2)\right] J_m(z) = 0$$

$$J_m(z) = \left(\frac{z}{2}\right)^p \left(1 + a_m z^2 + b_m z^4 + \cdots\right) \qquad (1)$$

$$\frac{d}{dz} J_m = \frac{p}{2}\left(\frac{z}{2}\right)^{p-1}\left(1 + a_m z^2 + b_m z^4 + \cdots\right)$$
$$+ \left(\frac{z}{2}\right)^p \left(2a_m z + 4 b_m z^3 + \cdots\right)$$

$$\frac{d^2}{dz^2} J_m = \frac{p(p-1)}{4}\left(\frac{z}{2}\right)^{p-2}\left(1 + a_m z^2 + b_m z^4 + \cdots\right)$$
$$+ p\left(\frac{z}{2}\right)^{p-1}\left(2a_m z + 4b_m z^3 + \cdots\right)$$
$$+ \left(\frac{z}{2}\right)^p \left(2a_m + 12 b_m z^2 + \cdots\right)$$

$$z^2 \frac{d^2}{dz^2} J_m + z\frac{d}{dz} J_m + (z^2 - m^2) J_m = 0$$

$$\Rightarrow 0 = p(p-1)\left(\frac{z}{2}\right)^p\left(1 + a_m z^2 + b_m z^4 + \cdots\right)$$
$$+ 2p\left(\frac{z}{2}\right)^p\left(2a_m z^2 + 4b_m z^4 + \cdots\right)$$
$$+ \left(\frac{z}{2}\right)^p\left(2a_m z^2 + 12 b_m z^4 + \cdots\right)$$
$$+ p\left(\frac{z}{2}\right)^p\left(1 + a_m z^2 + b_m z^4 + \cdots\right)$$
$$+ \left(\frac{z}{2}\right)^p\left(2a_m z^2 + 4b_m z^4 + \cdots\right)$$

$$+ \left(\frac{z}{2}\right)^p \left(z^2 + a_m z^4 + b_m z^6 + \cdots\right)$$

$$- m^2 \left(\frac{z}{2}\right)^p \left(1 + a_m z^2 + b_m z^4 + \cdots\right)$$

$$0 = \left(\frac{z}{2}\right)^p \Big\{ \left(p(p-1) + p - m^2\right)$$

$$+ \left(p(p-1) a_m + 4p a_m + 2a_m + p a_m + 2a_m + 1 - m^2 a_m\right) z^2$$

$$+ \left(p(p-1) b_m + 8p b_m + 12 b_m + p b_m + 4 b_m + a_m - m^2 b_m\right) z^4$$

$$+ \vdots$$

$$\Big\}$$

The series must vanish term by term

$O(z^0) \Rightarrow \quad p(p-1) + p - m^2 = 0$

$\Rightarrow \quad p^2 = m^2$

$\Rightarrow \quad \boxed{p = m}$ b/c we want regular solutions at $z = 0$.

$O(z^2) \Rightarrow \left(p(p-1) + 5p + 4 - m^2\right) a_m = -1$

$\Rightarrow \quad (\cancel{p^2} + 4p + 4 - \cancel{m^2}) a_m = -1$

$$\boxed{a_m = \frac{-1}{4(m+1)}}$$

$$O(z^4) \Rightarrow \left(p(p-1) + 9p + 16 - m^2\right) b_m = -a_m$$

$$\left(p^2 + 8p + 16 - \cancel{m^2}\right) b_m = -a_m$$

$$\boxed{b_m = \frac{-a_m}{8(m+2)} = \frac{1}{32(m+1)(m+2)}}$$

In general,

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+m+1)} \left(\frac{z}{2}\right)^n \qquad (2)$$

$$\Rightarrow \quad C_n = \frac{(-1)^n}{n!\,\Gamma(n+m+1)}$$

Let $(\alpha)_k = (\alpha)(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)$

$$= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)},$$

then $\Gamma(n+m+1) = (m+1)_n \Gamma(m+1)$

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(m+1)_n\,\Gamma(m+1)} \left(\frac{z}{2}\right)^{2n}$$

$$= \frac{1}{\Gamma(m+1)} \left(\frac{z}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(m+1)_n} \left(\frac{z}{2}\right)^{2n}$$

Up to the overall multiplicative factor $\frac{1}{\Gamma(m+1)}$,

$$C_n^m = \frac{(-1)^n}{n!\,(m+1)_n}$$

$\Rightarrow$ $C_0^m = 1$, (The Pochhammer symbol $(\alpha)_k$ for $k=0$ is 1)

$$C_1^m = \frac{-1}{m+1}$$

$$C_2^m = \frac{1}{2(m+1)(m+2)}$$

These agree with a_m and b_m up to the overall multiplicative factor $\frac{1}{\Gamma(m+1)}$.

<u>Note</u> that we've absorbed the factor of $\left(\frac{1}{2}\right)^n$ into z in eqn. (2) compared to eqn. (1).

(b) $J_m(z) = f(z)/\sqrt{z}$

$$\frac{d}{dz} J_m = \frac{1}{\sqrt{z}} f'(z) - \frac{1}{2} f(z)/\sqrt{z^3}$$

$$\frac{d^2}{dz^2} J_m = \frac{1}{\sqrt{z}} f''(z) - \frac{1}{\sqrt{z^3}} f'(z) - \frac{1}{2}\left(\frac{-3}{2}\right) f(z)/\sqrt{z^5}$$

$$= \frac{1}{\sqrt{z}} f''(z) - \frac{1}{\sqrt{z}^3} f'(z) + \frac{3}{4} f(z)/\sqrt{z}^5$$

$$0 = z^2 \frac{d^2}{dz^2} J_m + z \frac{d}{dz} J_m + (z^2 - m^2) J_m(z)$$

$$\Rightarrow 0 = \sqrt{z}^3 f''(z) - \sqrt{z} f'(z) + \frac{3}{4\sqrt{z}} f(z)$$

$$+ \sqrt{z} f'(z) - \frac{1}{2\sqrt{z}} f(z) + \sqrt{z}^3 f(z) - m^2 \frac{f(z)}{\sqrt{z}}$$

$$\Rightarrow 0 = z^2 f''(z) - z\cancel{f'}(z) + \frac{3}{4} f(z) + z\cancel{f'}(z) - \frac{1}{2} f(z) + z^2 f(z)$$
$$- m^2 f(z)$$

$$\Rightarrow f''(z) + \frac{1}{z^2}\left(\frac{1}{4} - m^2\right) f(z) = -f(z) \qquad (1)$$

Schrodinger eqn. is

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi(x) = E\, \psi(x)$$

or $$\left[\frac{d^2}{dx^2} - \frac{2m}{\hbar^2} V(x)\right] \psi(x) = -\frac{2mE}{\hbar^2} \psi(x) \qquad (2)$$

Comparing (1) & (2), we see that by identifying

$$-\frac{2m}{\hbar^2} V(z) = \frac{1}{z^2}\left(\frac{1}{4} - m^2\right)$$

$$\Rightarrow V(z) = -\frac{\hbar^2}{2m}\left(\frac{1}{4} - m^2\right)\frac{1}{z^2}, \quad \text{and}$$

$$-\frac{2mE}{\hbar^2} = -1 \Rightarrow E = \frac{\hbar^2}{2m}, \quad \text{we get}$$

a schrodinger equation for f.

The potential $V(z) \propto \frac{1}{z^2} \to 0$ as $z \to \infty$.

At $z \to \infty$, $(1) \to f''(z) = -f(z) \qquad (3)$

$f(z) = A_m \cos(z + \phi_m)$ is a general solution

for (3)

$$\Rightarrow J_m(z) \sim \frac{f(z)}{z} = \frac{A_m \cos(z + \phi_m)}{\sqrt{z}} \quad \text{as } z \to \infty.$$

(c) We know that $\psi_{jm}(r,\phi) = N_{jm} e^{im\phi} J_m(k_{jm} r)$

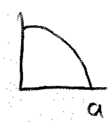
solves the schrodinger equation for a cylindrical square well.

Our intuition is that the ground state will have no nodes in the cylinder, and is even, whereas the 1^{st} excited state has 1 node, and is odd. Since we're in 2D, a node becomes a 1d line where the wave function is 0.

For $m=0$, $\psi_{j0} \propto J_0(k_{j0} r)$.

Setting $a=1$, we see that for $r<1$

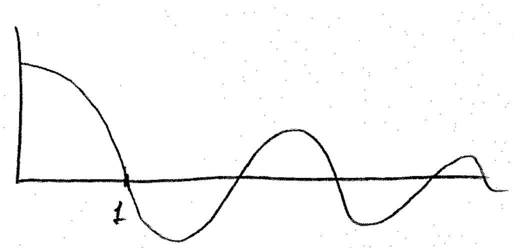
J_0 should have the shape.



At large r J_0 should go like $\frac{\cos(r)}{r}$

So,

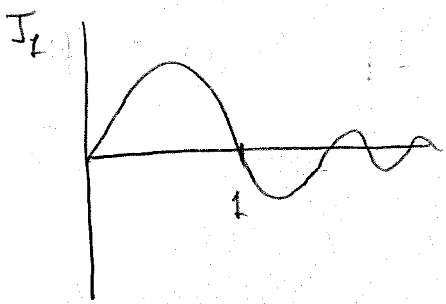
J_0



For J_1, we set $m=1$. ψ_{j1} should have 1 node, i.e. a line defined by $\phi = \phi^*$, where $\psi_{j1} = 0$

This is impossible unless $J_1 = 0$ at $r=0$.

So we can guess that J_1 has the following shape.



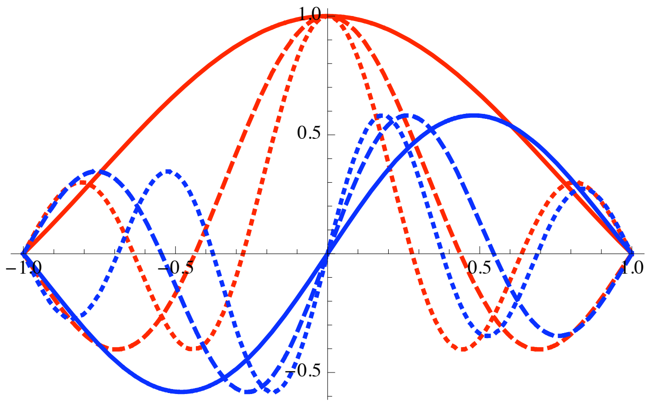
In fact, by the same argument, $J_m(0) = 0$ for any $m > 0$.

2. d.

Below I' ve plotted the first few wave functions.

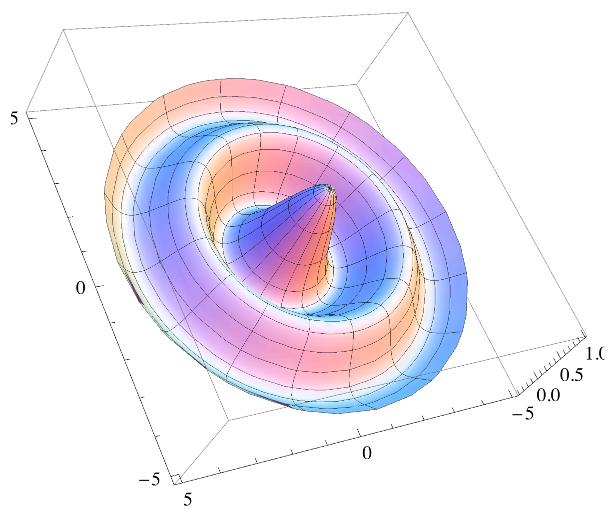
```
In[1583]:= Clear[k, m];
psi[r_] = If[r >= 0, BesselJ[m, k * Abs[r]], Cos[m π] * BesselJ[m, k * Abs[r]]];

Plot[{psi[r] /. {m → 0, k -> BesselJZero[0, 1]}, psi[r] /. {m → 0, k -> BesselJZero[0, 2]},
  psi[r] /. {m → 0, k -> BesselJZero[0, 3]}, psi[r] /. {m → 1, k -> BesselJZero[1, 1]},
  psi[r] /. {m → 1, k -> BesselJZero[1, 2]}, psi[r] /. {m → 1, k -> BesselJZero[1, 3]}},
 {r, -1, 1}, PlotStyle → {{Red, Thick}, {Red, Thick, Dashed}, {Red, Thick, Dotted},
   {Blue, Thick}, {Blue, Thick, Dashed}, {Blue, Thick, Dotted}}]
```



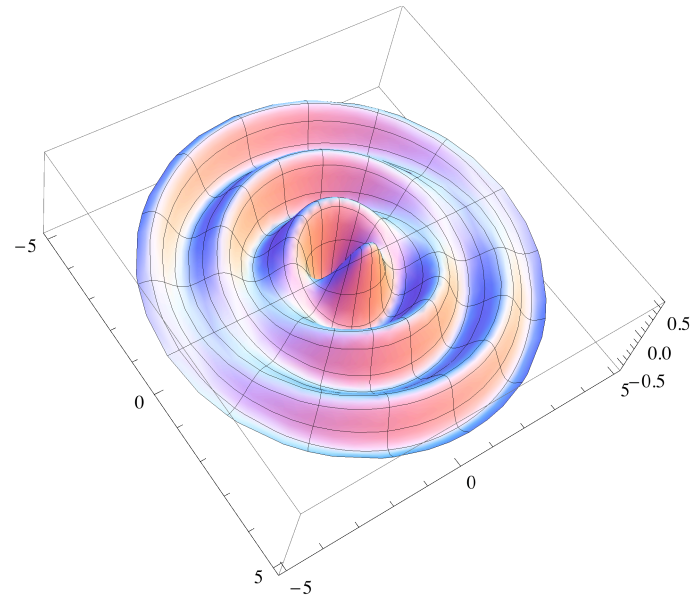
Plot of $\psi_{0,1}$

```
Clear[m, k];
psi2d[r_, p_] = Exp[ⅈ m p] * BesselJ[m, k * r];
RevolutionPlot3D[{Re[psi2d[r, p]] /. {m → 0, k → N[BesselJZero[0, 1]]}}, {r, 0, 5}, {p, 0, 2 π}]
```



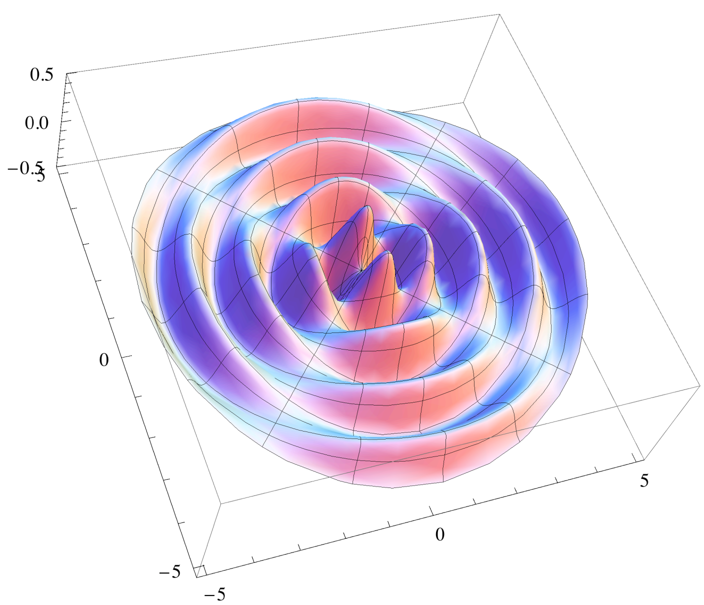
Plot of $\psi_{1,1}$

```
RevolutionPlot3D[{Re[psi2d[r, p]] /. {m → 1, k → N[BesselJZero[1, 1]]}}, {r, 0, 5}, {p, 0, 2 π}]
```



Plot of $\psi_{2,1}$

```
RevolutionPlot3D[{Re[psi2d[r, p]] /. {m → 2, k → N[BesselJZero[2, 1]]}}, {r, 0, 5}, {p, 0, 2 π}]
```



Zeros, k_{jm}

In[1575]:=
```
Clear[j, m, M];
M = {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}};
k = {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}};
For[m = 0, m < 4, m++,
 For[j = 1, j < 4, j++, {k[[m + 1, j]] = BesselJZero[m, j], M[[m + 1, j]] = N[BesselJZero[m, j]]}]]
Print[TraditionalForm[k], " = ", TraditionalForm[M]]
```

$$\begin{pmatrix} j_{0,1} & j_{0,2} & j_{0,3} \\ j_{1,1} & j_{1,2} & j_{1,3} \\ j_{2,1} & j_{2,2} & j_{2,3} \\ j_{3,1} & j_{3,2} & j_{3,3} \end{pmatrix} = \begin{pmatrix} 2.40483 & 5.52008 & 8.65373 \\ 3.83171 & 7.01559 & 10.1735 \\ 5.13562 & 8.41724 & 11.6198 \\ 6.38016 & 9.76102 & 13.0152 \end{pmatrix}$$

The energies are k_{jm}^2 in units of hbar ^ 2 / (2 m a ^ 2)

In[1580]:=
```
Clear[j, m, e];
e = {{E01, E02, E03}, {E11, E12, E13}, {E21, E22, E23}, {E31, E32, E33}};
For[m = 0, m < 4, m++, For[j = 1, j < 4, j++,
  {Print["E", m, j, " = ", N[BesselJZero[m, j]^2]], M[[m + 1, j]] = N[BesselJZero[m, j]^2]}]]
Print[TraditionalForm[e], " = ", TraditionalForm[M]]
```

E01 = 5.78319

E02 = 30.4713

E03 = 74.887

E11 = 14.682

E12 = 49.2185

E13 = 103.499

E21 = 26.3746

E22 = 70.85

E23 = 135.021

E31 = 40.7065

E32 = 95.2776

E33 = 169.395

$$\begin{pmatrix} E01 & E02 & E03 \\ E11 & E12 & E13 \\ E21 & E22 & E23 \\ E31 & E32 & E33 \end{pmatrix} = \begin{pmatrix} 5.78319 & 30.4713 & 74.887 \\ 14.682 & 49.2185 & 103.499 \\ 26.3746 & 70.85 & 135.021 \\ 40.7065 & 95.2776 & 169.395 \end{pmatrix}$$

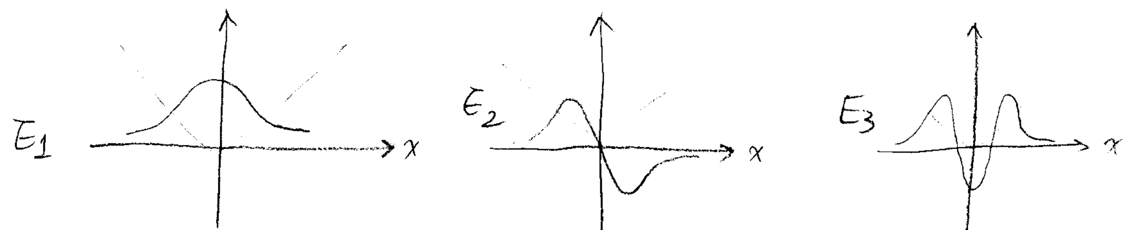
2. e.

So the lowest 6 eigenstates are 01, 11, 21, 02, 31, 12.

Phys 130 PS2 Solution

Problem 3

(a)



Notice that ① the wavefunctions must vanish as $x \to \pm\infty$

② ψ_1 and ψ_3 (corresponding to E_1 and E_3) are even, ψ_2 is odd.

(b) $$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + kx\,\psi(x) = E\,\psi(x) \quad (x>0).$$

$$x = c(z-a), \quad c = \sqrt[3]{\frac{\hbar^2}{2mk}}, \quad a = -\frac{E}{kc}$$

$$\Rightarrow \quad \frac{d^2\psi(z)}{dz^2} = z\,\psi(z)$$

(c). $Ai'(z)=0$: $z_1 = -1.02$, $z_2 = -3.25$, $z_3 = -4.82$

$Ai(z)=0$: $\tilde{z}_1 = -2.34$, $\tilde{z}_2 = -4.09$, $\tilde{z}_3 = -5.52$

(d) Since $Bi(z)$ blows up as $x \to \infty$, it cannot be a physical solution to the Schrödinger equation. On the other hand, $Ai(z)$ approaches zero as $x \to \infty$, so we must use only $Ai(z)$.

(e) even functions : $\psi'(x=0)=0$,

which means, $\psi'(z=a)=0$

Thus $a = z_1, z_2, z_3$.

$E_1 = (-kc)\cdot z_1 = 1.02\left(\frac{\hbar^2 k^2}{2m}\right)^{\frac{1}{3}}$, $\psi_1(x) = Ai\left(x\cdot\left(\frac{2mk}{\hbar^2}\right)^{\frac{1}{3}} + 1.02\right)$

$E_2 = (-kc)\cdot z_2 = 3.25\left(\frac{\hbar^2 k^2}{2m}\right)^{\frac{1}{3}}$, $\psi_2(x) = Ai\left(x\cdot\left(\frac{2mk}{\hbar^2}\right)^{\frac{1}{3}} + 3.25\right)$

$E_3 = (-kc)\cdot z_3 = 4.82\left(\frac{\hbar^2 k^2}{2m}\right)^{\frac{1}{3}}$, $\psi_3(x) = Ai\left(x\cdot\left(\frac{2mk}{\hbar^2}\right)^{\frac{1}{3}} + 4.82\right)$

(up to a normalization factor)

This is only for x>0. The wavefunction for x<0 can be obtained from \psi(x) = \psi(-x).

(f) odd functions $\tilde{\psi}(x=0)=0$,

which means, $\tilde{\psi}(z=a)=0$

Thus $a = \tilde{z}_1, \tilde{z}_2, \tilde{z}_3$,

$\tilde{E}_1 = 2.34\left(\frac{\hbar^2 k^2}{2m}\right)^{\frac{1}{3}}$, $\tilde{\psi}_1(x) = Ai\left(x\cdot\left(\frac{2mk}{\hbar^2}\right)^{\frac{1}{3}} + 2.34\right)$

$\tilde{E}_2 = 4.09\left(\frac{\hbar^2 k^2}{2m}\right)^{\frac{1}{3}}$, $\tilde{\psi}_2(x) = Ai\left(x\cdot\left(\frac{2mk}{\hbar^2}\right)^{\frac{1}{3}} + 4.09\right)$

$\tilde{E}_3 = 5.52\left(\frac{\hbar^2 k^2}{2m}\right)^{\frac{1}{3}}$, $\tilde{\psi}_3(x) = Ai\left(x\cdot\left(\frac{2mk}{\hbar^2}\right)^{\frac{1}{3}} + 5.52\right)$

$E_1 < \tilde{E}_1 < E_2 < \tilde{E}_2 < E_3 < \tilde{E}_3$ as desired.

This is only for x>0. The wavefunction for x<0 can be obtained from \psi(x) = -\psi(-x).