Problem Set 2 Solution

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Physics 130 Problem Set # 1

Solution of problem 1.

(a) If we define expectation values as

$$\langle A(x)\rangle = \int dx\; p(x)\; A(x)$$

then take $A(x) = 1$

$$1 = \int dx\; p(x).$$

(b) $\sigma^2 = \langle (x-\mu)^2\rangle = \langle x^2 - 2\mu x + \mu^2\rangle = \langle x^2\rangle - 2\mu\langle x\rangle + \langle \mu^2\rangle = \langle x^2\rangle - \langle x\rangle^2.$

(c) $\langle x\rangle = \int_0^1 dx\; p(x)\cdot x = \int_0^1 x\,dx = \boxed{\frac{1}{2}}.$

$\langle x^2\rangle = \int_0^1 dx \cdot x^2 = \frac{1}{3} \Rightarrow \sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = \boxed{\frac{1}{12}}.$

(d) It indeed makes sense. Assume the probability distribution is uniform over one strip, then

$$\sigma^2 = \frac{1}{12}$$

as we just computed in (c). Thus

$$\boxed{\sigma = \frac{1}{\sqrt{12}} < \frac{1}{2}}$$

(e) $\int_0^\infty A e^{-t/\tau}\,dt = 1 \Rightarrow \boxed{A = \frac{1}{\tau}}$

$\langle t\rangle = \int_0^\infty \frac{1}{\tau} e^{-t/\tau} t\,dt = \boxed{\tau},$

$\langle t^2\rangle = \int_0^\infty \frac{1}{\tau} e^{-t/\tau} t^2\,dt = 2\tau^2 \Rightarrow \sigma^2 = \langle t^2\rangle - \langle t\rangle^2 = \boxed{\tau^2}$

half-life time: $\int_0^{t_{1/2}} \frac{1}{\tau} e^{-t/\tau}\,dt = \frac{1}{2} \Rightarrow \boxed{t_{1/2} = \tau \ln 2 = \langle t\rangle \ln 2}.$

(f) $$\int_{-\infty}^{+\infty} B e^{-(x-a)^2/2\sigma^2}\, dx = 1.$$

$$\overset{y=x-a}{\Longrightarrow} \int_{-\infty}^{+\infty} B e^{-y^2/2\sigma^2}\, dy = 1.$$

$$\overset{z=y/\sqrt{2}\sigma}{\Longrightarrow} B\int_{-\infty}^{+\infty} e^{-z^2}\, dz \cdot \sqrt{2}\sigma = 1$$

$$\Rightarrow \boxed{B = \frac{1}{\sqrt{2\pi}\,\sigma}}$$

$$\langle x \rangle = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-a)^2/2\sigma^2} x\, dx$$

$$\overset{y=x-a}{=} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} (y+a)\, dy$$

$$= \frac{a}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{+\infty} e^{-y^2/2\sigma^2}\, dy$$

$$= \frac{a}{\sqrt{2\pi}\sigma} \cdot \sqrt{2\pi}\sigma$$

$$= \boxed{a}.$$

(g) $$f(x) \equiv e^{-\alpha x^2/2} x^n, \quad n \text{ is odd}.$$

Then $f(x) = -f(-x)$,

$$\int_{-\infty}^{+\infty} f(x)\, dx = \int_0^{\infty} f(x)\, dx + \int_{-\infty}^{0} f(x)\, dx$$

$$= \int_0^{\infty} f(x)\, dx + \int_{+\infty}^{0} f(-x)\, d(-x)$$

$$= \int_0^{\infty} f(x)\, dx + \int_0^{\infty} f(-x)\, dx$$

$$= 0.$$

For even n,

$$-\frac{\partial}{\partial\alpha} \int_{-\infty}^{+\infty} dx\, e^{-\alpha x^2/2} = -\frac{\partial}{\partial\alpha}\sqrt{\frac{2\pi}{\alpha}} = \frac{\sqrt{2\pi}}{2} \cancel{\frac{1}{\sqrt{2\pi}}}\, \alpha^{-\frac{3}{2}}.$$

Thus

$$\int_{-\infty}^{+\infty} dx\, e^{-\alpha x^2/2} \cdot \frac{x^2}{2} = -\frac{\partial}{\partial\alpha} \int_{-\infty}^{+\infty} dx\, e^{-\alpha x^2/2} = \frac{\sqrt{2\pi}}{2} \cancel{\frac{1}{\sqrt{2\pi}}}\, \alpha^{-\frac{3}{2}}$$

$$\Rightarrow \quad \int_{-\infty}^{+\infty} dx\, e^{-\alpha x^2/2} \cdot x^2 = \sqrt{2\pi}\,\alpha^{-\frac{3}{2}}.$$

Similarly,

$$-\frac{\partial}{\partial\alpha}\int_{-\infty}^{+\infty} dx \cdot x^2\, e^{-\alpha x^2/2} = -\frac{\partial}{\partial\alpha}\left(\sqrt{2\pi}\,\alpha^{-\frac{3}{2}}\right) = \frac{3}{2}\sqrt{2\pi}\,\alpha^{-\frac{5}{2}}$$

Thus

$$\int_{-\infty}^{+\infty} dx\, e^{-\alpha x^2/2}\, x^4 = 3\sqrt{2\pi}\,\alpha^{-\frac{5}{2}}$$

$\Rightarrow$ Generally,

$$\int_{-\infty}^{+\infty} dx\, e^{-\alpha x^2/2}\, x^n \overset{n \text{ even}}{=} \sqrt{2\pi}\, \underbrace{(n-1)!!}_{=1\cdot 3\cdot 5 \cdot \ldots \cdot (n-1)}\, \alpha^{-\frac{n+1}{2}}$$

(h) To compute variance, we need $\langle x\rangle$ and $\langle x^2\rangle$.

We've shown $\langle x\rangle = a$ in (f).

$$\langle x^2\rangle = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-a)^2/2\sigma^2} x^2\, dx$$

$$\overset{y=x-a}{=} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} (y+a)^2\, dy$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} y^2\, dy + a^2 \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \times \sqrt{2\pi}\left(\frac{1}{\sigma^2}\right)^{-\frac{3}{2}} + a^2$$

$$= \sigma^2 + a^2$$

Thus, variance $= \langle x^2\rangle - \langle x\rangle^2 = \sigma^2 + a^2 - a^2 = \boxed{\sigma^2}$

Similarly $\langle x^3\rangle \overset{y=x-a}{=} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} (y+a)^3\, dy$ $\qquad (y+a)^3 = \underbrace{\cancel{y^3}}_{n \text{ odd}} + 3y^2 a + \underbrace{\cancel{3a^2 y}}_{n \text{ odd}} + a^3$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} (3ay^2 + a^3)\, dy$$

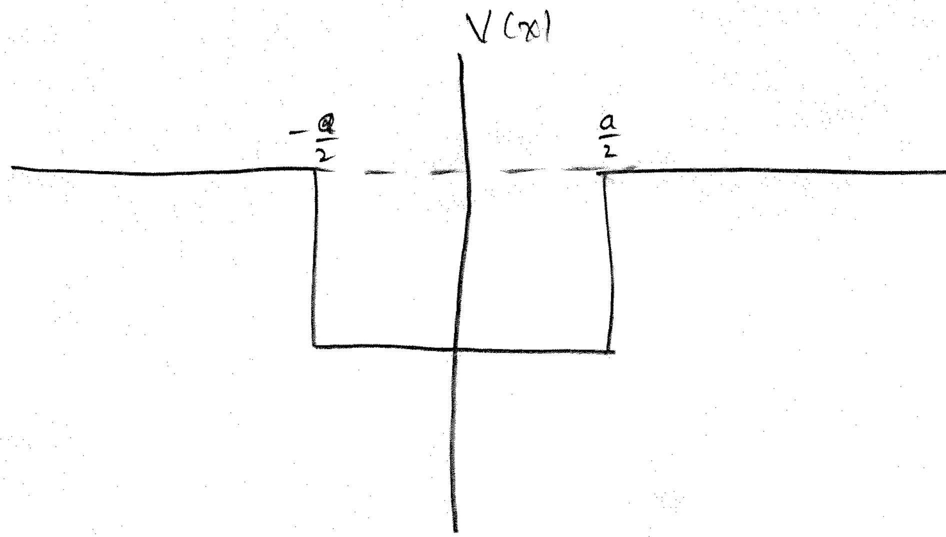
$$= \boxed{3a\sigma^2 + a^3}$$

$$\langle x^4 \rangle \overset{y=x-a}{=} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} (y+a)^4 \, dy \qquad (y+a)^4 = y^4 + \cancel{4y^3a} + 6a^2y^2 + \cancel{4ya^3} + a^4$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \left[\sqrt{2\pi} \cdot 3 \cdot \left(\frac{1}{\sigma^2}\right)^{-\frac{5}{2}} + 6a^2 \cdot \sqrt{2\pi} \left(\frac{1}{\sigma^2}\right)^{-\frac{3}{2}} + a^4 \cdot \sqrt{2\pi}\,\sigma \right]$$

$$= \boxed{3\sigma^4 + 6a^2\sigma^2 + a^4}.$$

2.



a) Assume the opposite, i.e. $\epsilon \geq W$, we will show that there are no sensible solutions to the schrodinger equation.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

$$\Rightarrow \frac{d^2}{dx^2}\psi(x) = -\frac{2m}{\hbar^2}\left[E - V(x)\right]\psi(x)$$

$$\Rightarrow \frac{d^2}{dx^2}\psi(x) = \frac{2m}{\hbar^2}\left[\epsilon + V(x)\right]\psi(x) \quad \text{since} \quad E = -\epsilon$$

For $\epsilon \geq W$, $\epsilon + V(x) > 0$ for all x.

If $\psi(x) > 0$, then $\frac{d^2}{dx^2}\psi(x) > 0$ also because

$$\frac{d^2}{dx^2}\psi(x) = \frac{2m}{\hbar^2}\underbrace{\left[\epsilon + V(x)\right]}_{+ve}\psi(x).$$

$\Rightarrow \psi(x)$ is concave up.

$\Rightarrow$ If $\psi(x) > 0$ at some x, then $\psi(x)$ is also concave up $\Rightarrow \frac{d\psi}{dx} > 0$. This holds for all x.

With this in mind let's begin to solve for ψ

I. The region $x < -\frac{a}{2}$

Here $V(x) = 0$

$$\Rightarrow \frac{d^2\psi}{dx^2} = \frac{2mE}{\hbar^2}\psi(x)$$

$$= \kappa^2 \psi(x), \quad \kappa = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\Rightarrow \psi(x) = Ae^{\kappa x} + Be^{-\kappa x}$$

Boundary condition, $\psi(x) \to 0$ as $x \to -\infty \Rightarrow B = 0$

$$\Rightarrow \psi(x) = Ae^{\kappa x}, \quad x < -\frac{a}{2}$$

Without loss of generality we may set A real and > 0. because if $\psi(x)$ is a solution to schrod. eqn., so if $C\psi(x)$ for any complex constant C.

$$\Rightarrow \psi(x) = e^{\kappa x} \;//$$

II. Region $x > \frac{a}{2}$

By similar reasoning we find that $\psi(x) = Ce^{\kappa x} + De^{-\kappa x}$

However, now we can not set $C = 0$, because that would mean that $\frac{d\psi}{dx} \sim -DK e^{-Kx} \to 0$ as $x \to \infty$, but earlier we have argued that $\psi(x)$ is concave up everywhere.

$\Rightarrow \psi(x) \sim Ce^{Kx}$ as $x \to \infty$, and this is not a sensible solution.

(b) $E < W$,

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}[E - V(x)]\,\psi(x)$$

I. $|x| > \frac{a}{2}$

Again, for $x < -\frac{a}{2}$, $\psi(x) = Ae^{Kx}$, $K = \sqrt{\frac{2m|W|}{\hbar^2}}$

for $x > \frac{a}{2}$, $\psi(x) = Fe^{-Kx}$

II. $|x| < \frac{a}{2}$

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}[W - E]\,\psi(x) = -l^2\,\psi(x)$$

General solution, $\psi(x) = C\sin(lx) + D\cos(lx)$

(c) First we argue that ψ should be continuous. We use physical argument. If ψ is discontinuous at some point x^*, then $\frac{d\psi}{dx}$ diverges $\Rightarrow$ the momentum, $p = -i\frac{d}{dx}$ diverges, but since $E = \frac{p^2}{2m}$, and E is assumed to be finite, we run into trouble. $\therefore$ $\psi(x)$ is continuous.

To argue that $\frac{d\psi}{dx}$ should be continuous, start with the t-independent schrodinger equation

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}\left(E - V(x)\right)\psi(x). \qquad (1)$$

Integrate (1) over a tiny region centered around an arbitrary $x = x^*$.

$$\int_{x^*-\frac{\varepsilon}{2}}^{x^*+\frac{\varepsilon}{2}} \frac{d^2\psi}{dx^2} = \frac{-2m}{\hbar^2}\int_{x^*-\frac{\varepsilon}{2}}^{x^*+\frac{\varepsilon}{2}} \left(E - V(x)\right)\psi(x)$$

As $\varepsilon \to 0$, we get,

$$\text{LHS} = \left.\frac{d\psi}{dx}\right|_{x^*_+} - \left.\frac{d\psi}{dx}\right|_{x^*_-}$$

$$\text{RHS} = \frac{-2m}{\hbar^2}\left\{ (E - V(x))\psi(x)\Big|_{x^*_+} - (E - V(x))\psi(x)\Big|_{x^*_-} \right\} \varepsilon$$

$$= \frac{-2m}{\hbar^2}\left\{ E\left(\psi(x)\Big|_{x^*_+} - \psi(x)\Big|_{x^*_-} \right)\varepsilon \right.$$

$$\left. + \left(V(x)\psi(x)\Big|_{x^*_+} - V(x)\psi(x)\Big|_{x^*_-} \right)\varepsilon \right\}$$

Since ψ is continuous, $\psi(x)\Big|_{x^*_+} - \psi(x)_{x^*_-}$ is finite

$\Rightarrow \left(\psi(x)\Big|_{x^*_+} - \psi(x)_{x^*_-} \right)\varepsilon \to 0$ as $\varepsilon \to 0$.

Similarly, if $V(x)$ is finite over $(x^* - \delta, x^* + \delta)$, $\delta > \varepsilon$,

$V(x)\psi(x)\Big|_{x^*_+} - V(x)\psi(x)\Big|_{x^*_-}$ is finite

$\Rightarrow \left(V(x)\psi(x)\Big|_{x^*_+} - V(x)\psi(x)\Big|_{x^*_-} \right)\varepsilon \to 0$ as $\varepsilon \to 0$

$\therefore$ RHS $\to 0$ as long as $V(x)$ is finite.

$\Rightarrow \frac{d\psi}{dx}\Big|_{x^*_+} - \frac{d\psi}{dx}\Big|_{x^*_-} \to 0 \Rightarrow \frac{d\psi}{dx}$ is continuous

at x^*.

In our problem, $V(x)$ is finite everywhere. $\therefore$ $\psi(x)$ & $\psi'(x)$ are everywhere continuous. $\boxtimes$

(d) So far we've found

$$\psi(x) = \begin{cases} Ae^{kx} & , x < -\frac{a}{2} \\ C\sin(lx) + D\cos(lx) & , -\frac{a}{2} < x < \frac{a}{2} \\ Fe^{-kx} & , x > \frac{a}{2} \end{cases}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \quad , \quad l = \sqrt{\frac{2m}{\hbar^2}(|V| - E)}$$

For even solutions, let's set $C = 0$, otherwise we have no hope of finding even solutions.

Then,

$$\psi(x) = \begin{cases} Ae^{kx} \\ D\cos(lx) \\ Fe^{-kx} \end{cases} \quad , \quad \psi'(x) = \begin{cases} Ake^{kx} \\ -Dl\sin(lx) \\ -kFe^{-kx} \end{cases}$$

I. <u>Continuity of ψ at $-\frac{a}{2}$</u>

$$Ae^{-k\frac{a}{2}} = D\cos\left(-\frac{la}{2}\right) = D\cos\left(\frac{la}{2}\right) \qquad (2)$$

II. <u>Continuity of ψ' at $-\frac{a}{2}$</u>

$$Ake^{-k\frac{a}{2}} = -Dl\sin\left(-\frac{la}{2}\right) = Dl\sin\left(\frac{la}{2}\right) \qquad (3)$$

III. Continuity of ψ at $\frac{a}{2}$

$$D\cos\left(\frac{la}{2}\right) = F e^{-\kappa a/2} \qquad (4)$$

IV. Continuity of ψ' at $\frac{a}{2}$

$$-lD\sin\left(\frac{la}{2}\right) = -\kappa F e^{-\kappa\frac{a}{2}} \qquad (5)$$

$$(2) \Rightarrow \quad A = D e^{\kappa\frac{a}{2}} \cos\left(\frac{la}{2}\right) \qquad (6)$$

$$(4) \Rightarrow \quad F = D e^{\kappa\frac{a}{2}} \cos\left(\frac{la}{2}\right) \qquad (7)$$

$$(6)\ \&\ (7) \Rightarrow \quad \boxed{A = F = D e^{\kappa\frac{a}{2}} \cos\left(\frac{la}{2}\right)} \qquad (8)$$

We have all the constants up to an overall constant. (D for example).

(e) $(3)/(2) \Rightarrow \kappa = l\tan\left(\frac{la}{2}\right)$, $(5)/(4)$ gives the same equation.

$$l^2 = \frac{2m}{\hbar^2}W - \frac{2mE}{\hbar^2} = \frac{2mW}{\hbar^2} - \kappa^2$$

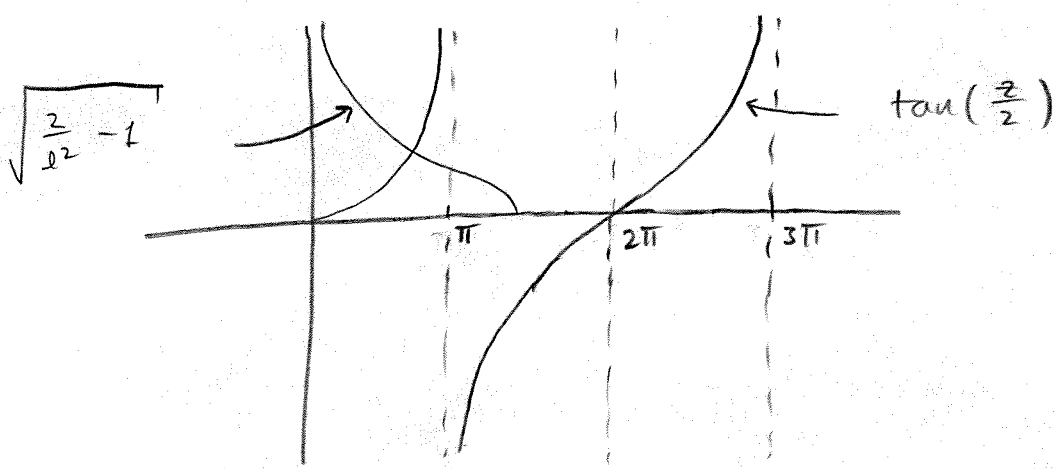
$$\Rightarrow \kappa = \sqrt{\frac{2mW}{\hbar^2} - l^2}$$

$$\Rightarrow \frac{1}{l}\sqrt{\frac{2mW}{\hbar^2} - l^2} = \tan\left(\frac{la}{2}\right)$$

$$\boxed{\sqrt{\frac{2mW}{\hbar^2 l^2} - 1} = \tan\left(\frac{la}{2}\right)} \qquad (9)$$

(f) let $la = z$, $\frac{2mWa^2}{\hbar^2} = z_0$, then

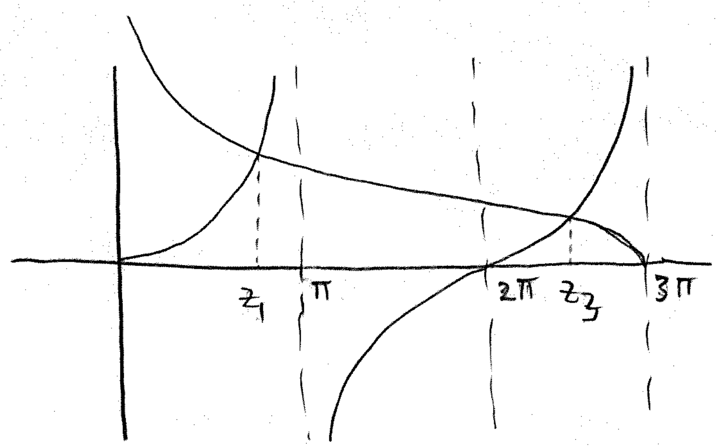
$$\text{eqn } (9) \rightarrow \sqrt{\frac{z_0^2}{z^2} - 1} = \tan\left(\frac{z}{2}\right)$$



For $W = \frac{9\hbar^2\pi^2}{2ma^2}$, $\frac{2mWa^2}{\hbar^2} = \frac{2ma^2}{\cancel{\hbar^2}}\left(\frac{9\pi^2\cancel{\hbar^2}}{2ma^2}\right) = (3\pi)^2$

$$(9) \Rightarrow \sqrt{\frac{(3\pi)^2}{z^2} - 1} = \tan\frac{z}{2}$$

LHS $= 0$ at $z = 3\pi$



The two solutions, z_1 and z_3, can be found using mathematica.

The corresponding energies are given by $W + E_{1,3} = \frac{z_{1,3}^2 \hbar^2}{2ma^2}$

$$\Rightarrow E_{1,3} = \left(-9 + \frac{z_{1,3}^2}{\pi^2}\right) \frac{\pi^2 \hbar^2}{2ma^2}$$

Using Mathematica, (see attached) we get
$$\boxed{\begin{aligned} E_1 &= -8.323 \frac{\pi^2\hbar^2}{2ma^2} \\ E_3 &= -3.206 \frac{\pi^2\hbar^2}{2ma^2} \end{aligned}}$$

(g) For any $W > 0$, the LHS eqn. (9) hits 0 for some positive value of l, as we can see from eqn. (10). This implies that there is always at least one intersection of the two curves.

(h) From the graph, we see that as W gets big, the curve of the LHS of eqn. (9) stretches ever more to the right, and the intersections approach the values

$$z_n \to (2n+1)\pi \quad \text{from the left.}$$

$$\Rightarrow z_n^2 \cong (2n+1)^2 \pi^2 = \left(\frac{2mW}{\hbar^2} - \frac{2mE_n}{\hbar^2}\right) a^2 = \frac{2ma^2}{\hbar^2}(W + E_n)$$

$$\Rightarrow \quad W + E_n \cong (2n+1)^2 \frac{\pi^2 \hbar^2}{2ma^2}$$

$\therefore$ The energy above the bottom of the well approaches the energies of the even infinite square well.

(i) For the odd solutions, let's set $D=0$ in the solution we found for $\psi(x)$ in (d).

Then,
$$\psi(x) = \begin{cases} A e^{\kappa x} & , x < -\frac{a}{2} \\ C\sin(lx) & , -\frac{a}{2} < x < \frac{a}{2} \\ F e^{-\kappa x} & , x > \frac{a}{2} \end{cases}$$

$$\psi'(x) = \begin{cases} A\kappa e^{\kappa x} \\ Cl\cos(lx) \\ -\kappa F e^{-\kappa x} \end{cases}$$

I. <u>Continuity of ψ at $-\frac{a}{2}$</u>

$$A e^{-\kappa a/2} = C\sin\left(-\frac{la}{2}\right) = -C\sin\left(\frac{la}{2}\right) \qquad (11)$$

II. <u>Continuity of ψ' at $-\frac{a}{2}$</u>

$$A\kappa e^{-\kappa \frac{a}{2}} = Cl\cos\left(-\frac{la}{2}\right) = Cl\cos\left(\frac{la}{2}\right) \qquad (12)$$

$(11) \Rightarrow \quad A = -C e^{\kappa a/2} \sin\left(\frac{la}{2}\right)$, fixing C fixes A.

$(11)/(12) \Rightarrow \quad \frac{1}{\kappa} = -\frac{1}{l} \tan\left(\frac{la}{2}\right)$

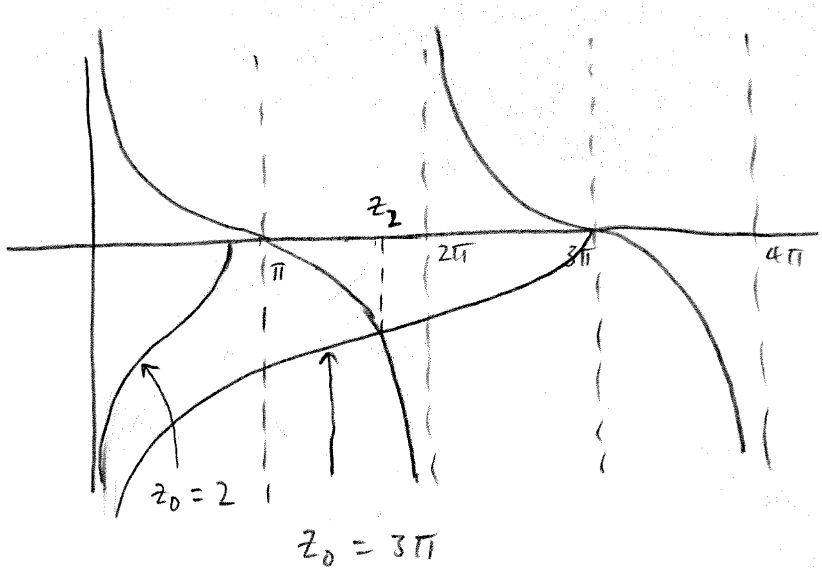
$$-\frac{l}{\kappa} = \tan\left(\frac{la}{2}\right)$$

$$-\frac{\kappa}{l} = \cot\left(\frac{la}{2}\right)$$

$$-\sqrt{\frac{2mWa^2}{\hbar^2 (la)^2} - 1} = \cot\left(\frac{la}{2}\right)$$

let $\frac{2mWa^2}{\hbar^2} = z_0^2$, $la = z$,

$$-\sqrt{\frac{z_0^2}{z^2} - 1} = \cot\left(\frac{z}{2}\right) \qquad (13)$$



LHS of eqn. (13) hits 0 when $z = z_0$

We can see from the graph that unless $z_0 > \pi$, there are no solutions.

For $W = \frac{9\hbar^2\pi^2}{2ma^2} \Rightarrow z_0^2 = \frac{2ma^2}{\hbar^2}\left(\frac{9\hbar^2\pi^2}{2ma^2}\right) = (3\pi)^2$,

$$(13) \rightarrow \quad -\sqrt{\frac{(3\pi)^2}{z^2} - 1} = \cot\left(\frac{z}{2}\right)$$

We can find the intersection at $z = z_2$ numerically using mathematica (see attached).

The corresponding value of the energy is

$$z_2^2 = (l_2 a)^2 = a^2 \left\{ \frac{2m}{\hbar^2} W - \frac{2m E_1}{\hbar^2} \right\}$$

$$\Rightarrow \frac{2ma^2}{\hbar^2}(W + E_2) = z_2^2$$

$$\boxed{W + E_2 = \frac{z_2^2 \hbar^2}{2ma^2}} \qquad (14)$$

$$\boxed{E_2 = -6.332 \frac{\pi^2 \hbar^2}{2ma^2}}$$

(j) As $W \to \infty$, $z_0 \to \infty$, and the intersections occur near

$z_n = 2n\pi$. From (14),

$$\Rightarrow \quad W + E_n \cong \frac{(2n)^2 \pi^2 \hbar^2}{2ma^2}$$

(k)
$$\psi_n(x) = \begin{cases} D_n \cos\left(\frac{l_n a}{2}\right) e^{k_n\left(\frac{a}{2}+x\right)} & , \ x < -\frac{a}{2} \\ D_n \cos(l_n x) & , \ -\frac{a}{2} \leq x \leq \frac{a}{2} \\ D_n \cos\left(\frac{l_n a}{2}\right) e^{-k_n\left(-\frac{a}{2}+x\right)} & , \ \frac{a}{2} < x \end{cases}$$

for n even,

$$\psi_n(x) = \begin{cases} -C_n \sin\left(\frac{l_n a}{2}\right) e^{k_n\left(\frac{a}{2}+x\right)} \\ C_n \sin(l_n x) \\ C_n \sin\left(\frac{l_n a}{2}\right) e^{-k_n\left(-\frac{a}{2}+x\right)} \end{cases}$$

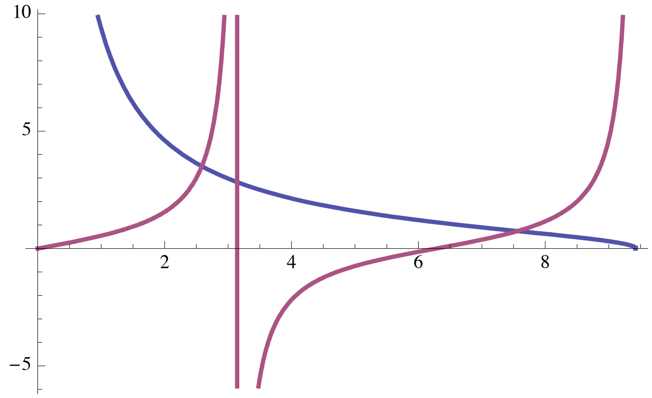
C_n & D_n are determined by normalization. For now let's set them to 1.

(f)

In[1177]:= **Clear[a];**

In[1212]:= **Plot[{ Sqrt[(3 π / z)^2 - 1], Tan[z / 2]}, {z, 0, 3 π}, PlotStyle → Thick]**

Out[1212]=



In[1186]:= **z1 = FindRoot[Tan[z / 2] - Sqrt[(3 π / z)^2 - 1], {z, 2}]**
z3 = FindRoot[Tan[z / 2] - Sqrt[(3 π / z)^2 - 1], {z, 7}]

Out[1186]= {z → 2.58575}

Out[1187]= {z → 7.56224}

In[1207]:= **E1 = (-9 + z^2 / π^2) /. z1**

Out[1207]= -8.32256

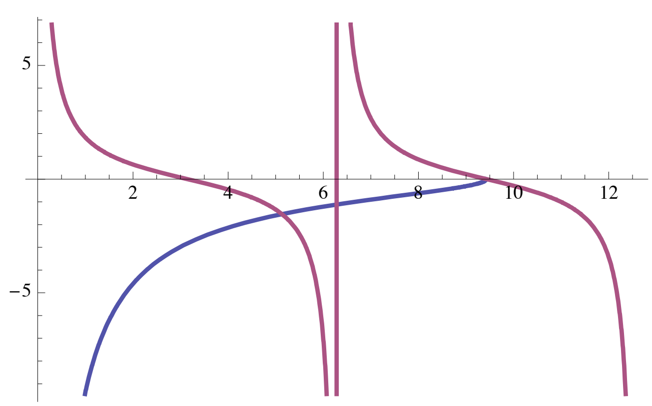
In[1209]:= **E2 = (-9 + z^2 / π^2) /. z3**

Out[1209]= -3.2057

(g)

In[1211]:= **Plot[{-Sqrt[(3 π)^2 / z^2 - 1], Cot[z / 2]}, {z, 0, 4 π}, PlotStyle → Thick]**

Out[1211]=



In[1200]:= **z2 = FindRoot[Cot[z / 2] + Sqrt[(3 π / z)^2 - 1], {z, 1}]**

Out[1200]= {z → 5.13164}

In[1210]:=
```
E3 = (-9 + z^2 / π^2) /. z2
```

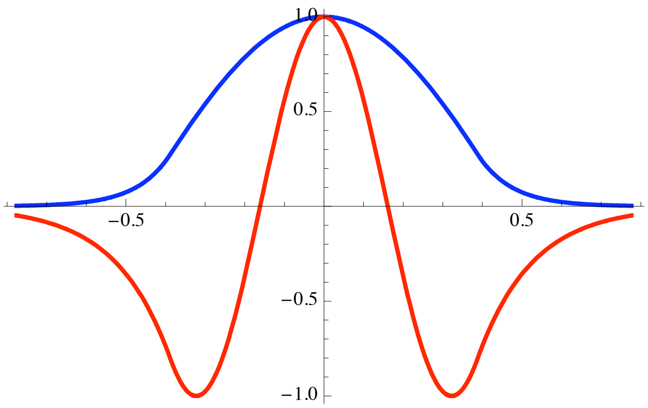
Out[1210]=
```
-6.33184
```

(k)

In[1201]:=
```
a = RandomReal[];
psieven[x_] = If[x < -a / 2, Cos[z / 2] Exp[Sqrt[(3 π / a)^2 - (z / a)^2] (a / 2 + x)],
   If[x > a / 2, Cos[z / 2] Exp[-Sqrt[(3 π / a)^2 - (z / a)^2] (-a / 2 + x)], Cos[z / a x]]];
```

In[1203]:=
```
ploteven = Plot[{psieven[x] /. z1, psieven[x] /. z3},
  {x, -a, a}, PlotStyle → {{Thick, Blue}, {Thick, Red}}]
```

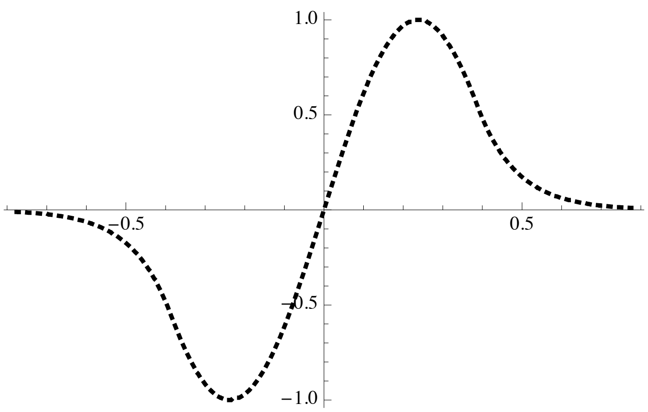
Out[1203]=



In[1204]:=
```
psiodd[x_] = If[x < -a / 2, -Sin[z / 2] Exp[Sqrt[(3 π / a)^2 - (z / a)^2] (a / 2 + x)],
   If[x > a / 2, Sin[z / 2] Exp[-Sqrt[(3 π / a)^2 - (z / a)^2] (-a / 2 + x)], Sin[z / a x]]];
```

In[1205]:=
```
plotodd = Plot[psiodd[x] /. z2, {x, -a, a}, PlotStyle → {Thick, Black, Dotted}]
```

Out[1205]=



In[1206]:= `Show[ploteven, plotodd]`

Out[1206]=