Math Diagnostic Solution

January 24, 2013

"Closed Book" - Please limit time to ~75 min.

1- What is the magnitude $|z|$ of the following complex number: $z=3-4i$?

$|z| = \sqrt{3^2+4^2} = 5$

1b- Did you learn about complex numbers in a previous course? Yes No

1c- Was problem 1 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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2- What is the real part of the expression $re^{i\cos\theta}$ (where r and θ are real constants)?

$\mathrm{Re}\{r[\cos(\cos\theta) + i\sin(\cos\theta)]\} = r\cos(\cos\theta)$

4

2b- Was problem 2 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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3- Which of the following is a possible solution to $5\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 0$?

(a) $Ae^{2t/5} + Be^{-2t/5}$ X

(b) $Ae^{-2t/5}\cos\left(\frac{t}{5}\right)$

(c) $A\cos\left(\frac{t}{5}\right) + B\sin\left(\frac{t}{5}\right)$ X

(d) $Ae^{-2t/5} + Bte^{-2t/5}$ X

(e) $A\cosh\left(\frac{t}{5}\right) + B\sinh\left(\frac{t}{5}\right)$

3b- Was problem 3 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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4- What is the magnitude $|f(z)|$ of the following complex function: $f(z) = 5ize^{i\pi/3}$?

$$\sqrt{(5ize^{i\pi/3})(-5ize^{-i\pi/3})} = 5|z|$$

4b- Was problem 4 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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5- Evaluate $\int_0^{4\pi} 3e^{i(x+\tan\frac{\pi}{3})}\,dx$

$$= 3e^{i\tan\frac{\pi}{3}} \int_0^{4\pi} e^{ix}\,dx$$

$$= 3e^{i\tan\frac{\pi}{3}} \left[\frac{1}{i} e^{ix}\right]_0^{4\pi}$$

$$= 3e^{i\tan\frac{\pi}{3}} (-i)\left(e^{4\pi i} - 1\right)$$

$$= 0$$

5b- Did you learn about this in a previous course? Yes No

5c- Was problem 5 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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6- A new grocery store just opened in the middle of town. In order to attract customers they hold an original promotion where each customer can save money on their purchase by rolling a traditional six face die -- the higher the outcome, the larger the savings. **For instance, if the customer rolls a "6"or a "5", he (or she) saves 4% of the total purchase amount, a roll of "4" leads to a saving of 3%, a roll "3" or "2" leads to a savings of 2%, while a roll of "1", leads to a saving of 1%.** Which of the following percentages is closest to the expected average savings for the customer?"

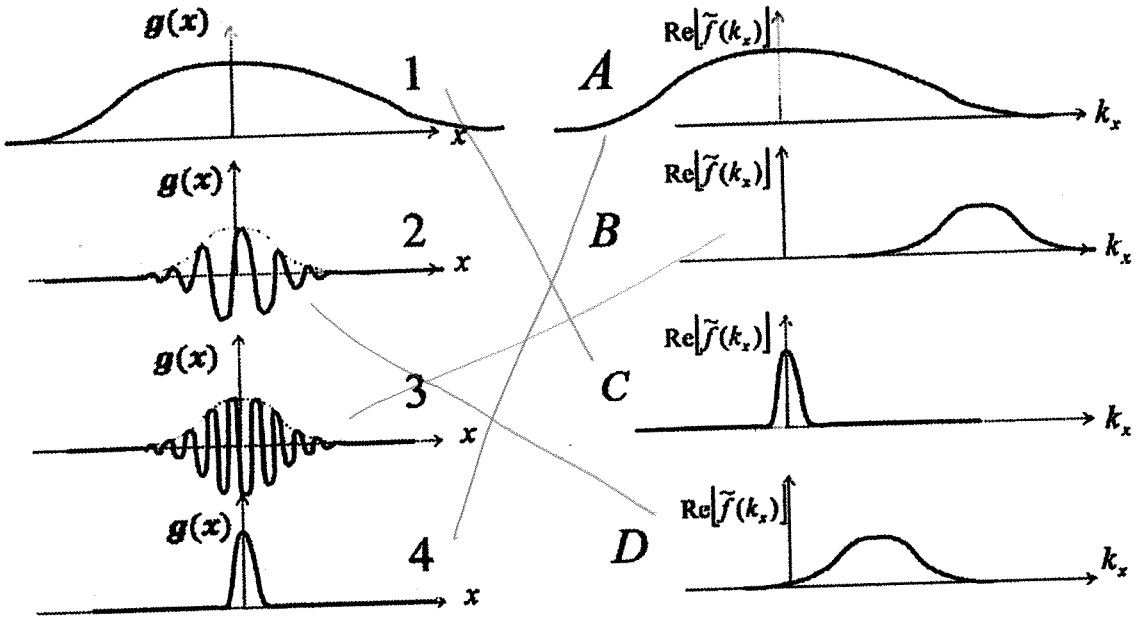
(a) 1.5% (b) 2.0% (c) 2.5% (d) 3.5% (e) 16.0%

$$\frac{1}{3}(4\%) + \frac{1}{6}(3\%) + \frac{1}{3}(2\%) + \frac{1}{6}(1\%)$$

$$= \frac{4}{3}\% + \frac{1}{2}\% + \frac{2}{3}\% + \frac{1}{6}\% = \frac{8+3+4+1}{6} = \frac{16}{6} = 2.6\%$$

6c- Was problem 6 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

7- Match the real part of the real function $g(x)$ to the real part of its Fourier Transform, denoted as $\mathrm{Re}(\tilde{f}(k_x))$.



7b- Did you learn about Fourier transforms in a previous course? Yes No

If YES, which course(s)?

7c- Was problem 7 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

8- Can you approximate $f(x) = 5(1-x)^7$ as a linear expression of the form $f(x) \approx a + bx$ (assuming $|x| \ll 1$)? If you can, what are the values for a and b?

$f(x) \approx 5\left[1^7 - \binom{7}{1}(1)^6 x + \cdots\right] = 5(1-7x)$

$= 5 - 35x$

a b

8b- Did you learn about this in a previous course? Yes No

8c- Was problem 8 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

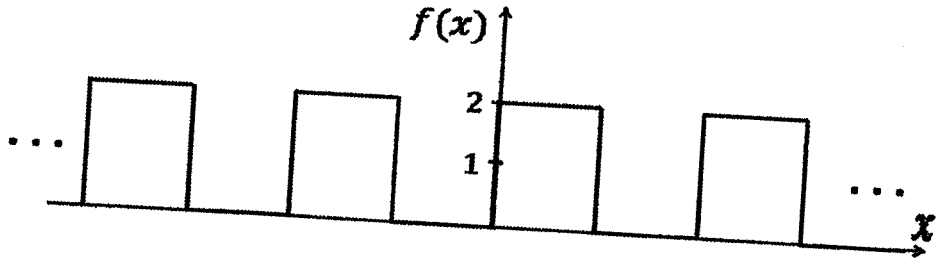
9- Evaluate $\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{(x^2+y^2)}{2\sigma^2}}\right)$ (σ is a constant).

$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(\frac{x^2+y^2}{2\sigma^2}\right)} \frac{\partial}{\partial x}\left(\frac{x^2+y^2}{2\sigma^2}\right)$

$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{x^2+y^2}{2\sigma^2}} \left(\frac{x}{\sigma^2}\right)$

9b- Was Q9 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

10- Which Fourier series is the correct expansion for the function $f(x)$?



(a) $f(x) = 1 + \frac{4}{\pi}sin\pi x + \frac{4}{\pi}cos\pi x + \frac{4}{3\pi}sin3\pi x + \cdots$

(b) $f(x) = \frac{4}{\pi}sin\pi x + \frac{4}{3\pi}sin3\pi x + \frac{4}{5\pi}sin5\pi x + \frac{4}{7\pi}sin7\pi x + \cdots$

(c) $f(x) = 1 + \frac{4}{\pi} sin\pi x + \frac{4}{3\pi} sin3\pi x + \frac{4}{5\pi} sin5\pi x + \cdots$

(d) $f(x) = \frac{4}{\pi} sin\pi x + \frac{4}{\pi} cos\pi x + \frac{4}{3\pi} sin3\pi x + \frac{4}{3\pi} cos3\pi x + \cdots$

(e) $f(x) = 1 + \frac{4}{\pi} cos\pi x + \frac{4}{3\pi} cos3\pi x + \frac{4}{5\pi} cos5\pi x + \cdots$

10b- Did you learn about Fourier series in a previous course? Yes No

10c- Was Q10 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

11- What is the shortest distance x over which the function $f(x) = \cos\left[3\left(\frac{\pi x}{2p} - \frac{\pi}{3}\right)\right]$ repeats itself (i.e. wavelength) ?

(a) $p/6$ (b) p (c) $p/3$

(d) $4p/3$ (e) $p/3\pi$ (f) $p/6\pi$

11b- Was Q11 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

12- How many linearly independent solutions does $5\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + y = 0$ have, (i.e. $y_1(t), y_2(t), y_3(t), y_4(t), \ldots$) ?

(a) It depends on the initial conditions: $y(0), y'(0)$...

(b) 2

(c) 4

(d) 1

(e) 3

12b- Did you learn about this in a previous course? Yes No

12c- Was Q12 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

13- Two traveling waves are described by the equations: $Y_1(x,t) = 8\cos(4kx + 2\omega t)$ and $Y_2(x,t) = 2\cos(2kx + \omega t)$. Which wave has the highest propagation speed?

(a) Wave 1

(b) Wave 2

(c) Both waves have the same speed

13b- Was Q13 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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14- Given the solution to a linear differential equation $y(x) = A\sinh(kx) + B\cosh(kx)$, the boundary condition $y'(0) = 0$ implies:

$y'(x) = A\cosh(kx)k + Bk\sinh(kx)$

$y'(0) = A\cosh(0)k + 0$

$\Rightarrow A = 0$

(a) $A = 0$ (b) $B = 0$

(c) $k = 0$ (d) $k = n\pi,\ n = 1,2,3\ldots$

(e) A trivial solution for all x (i.e. $y(x) = 0$)

14b- Was Q14 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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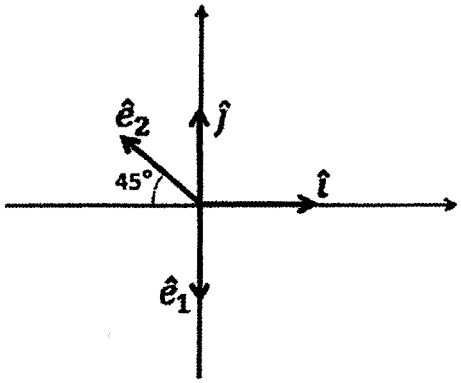
15- The velocity of a particle is given by $\vec{v}(t) = v_0(\sin(t)\,\hat{\imath} + \cos(t)\hat{\jmath})$. Express the velocity $\vec{v}(t)$ in terms of the unit vectors $\hat{e}_1$ and $\hat{e}_2$:



$\hat{e}_2 = N(\hat{j} - \hat{i})$ $\quad \hat{e}_2 \cdot \hat{e}_2 = 1 \Rightarrow N^2(1+1) \Rightarrow N^2 = \frac{1}{2}$, $N = \frac{1}{\sqrt{2}}$

$\hat{e}_2 = \frac{1}{\sqrt{2}}(\hat{j} - \hat{i})$ $\quad \hat{i} = \hat{j} - \sqrt{2}\hat{e}_2$

$\hat{e}_1 = -\hat{j}$ $\quad = -\hat{e}_1 - \sqrt{2}\hat{e}_2$

$\Rightarrow \hat{j} = -\hat{e}_1$ $\Rightarrow \vec{v}(t) = v_0\{\sin t(-\hat{e}_1 - \sqrt{2}\hat{e}_2) + \cos t(-\hat{e}_1)\}$

$\vec{v}(t) = v_0\left(-(\sin t + \cos t)\hat{e}_1 - \sqrt{2}\sin t\,\hat{e}_2\right)$

$= -v_0\left[(\sin t + \cos t)\hat{e}_1 + \sqrt{2}\sin t\,\hat{e}_2\right]$

15b- Was Q15 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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16- Evaluate $\int_{z=0}^{z=L}\int_{y=0}^{y=1/\sqrt{2}}\int_{x=0}^{x=y} 2z\,dxdydz$:

$= \int_{z=0}^{L} dz\,2z\left\{\int_0^{1/\sqrt{2}} dy\left(\int_0^{y} dx\right)\right\}$

$= \int dz\,2z\left\{\int_0^{1/\sqrt{2}} dy\,y\right\} = \int dz\,2z\left[\frac{y^2}{2}\right]_0^{1/\sqrt{2}} = \int_0^{L} dz\,2z\left(\frac{1}{2}\cdot\frac{1}{2}\right)$

$= \int_0^{L} dz = L$ //

16b- Did you learn about this in a previous course? Yes No

16c- Was Q16 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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17- Large forest fires (or any large fire for that matter) are known to produce hurricane force winds in their vicinity. Large temperature gradients drive air masses at spectacular speeds. What mathematical expression is appropriate to describe the density of air (kg/m^3) in and around the fire?

(a) A scalar field $\rho = \rho(x, y, z, t)$ Density is a scalar.

(b) A vector field $\vec{\rho} = \left(\rho_x(x, y, z, t), \rho_y(x, y, z, t), \rho_z(x, y, z, t)\right)$

(c) In this particular case, the density ρ cannot be expressed as a scalar or a vector field.

Please comment on your choice:

17b- Did you learn about this in a previous course? Yes No

17c- Was Q17 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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18- The matrix $\boldsymbol{R} = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$ takes a vector $\boldsymbol{r} = (r_x, r_y, r_z)$ and rotates it by an angle θ about an axis. What is the axis of rotation? *(assume that we are looking down the positive axis of rotation toward the origin)*

for $\theta = \frac{\pi}{2}$, $R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

take $\vec{r} = \hat{x}$

$R\hat{x} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -\hat{z}$

$\Rightarrow$ (d).

(a) x-axis -- clockwise (b) x-axis -- counterclockwise

(c) y-axis -- clockwise (d) y-axis -- counterclockwise

(e) z-axis -- clockwise (f) z-axis -- counterclockwise

18b- Did you learn about this in a previous course? Yes No

18c- Was Q18 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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19- The vector $\boldsymbol{v} = (v_x, v_y, v_z)$ is an eigenvector of the matrix

$$\boldsymbol{R} = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$ (same rotation matrix as in Prob. 18).

Which of the vectors below is a candidate for $\boldsymbol{v}$?

(a) $\boldsymbol{v} = (0,0,-1)$ ✓ (b) $\boldsymbol{v} = (-1,0,1)$

(c) $\boldsymbol{v} = (1,1,1)$ ✗ (d) $\boldsymbol{v} = (0,-1,0)$ $\vec{R}\vec{v} = \vec{v}$, with eigenvalue 1

(e) $\boldsymbol{v} = (1,0,-1)$ ✗ (f) $\boldsymbol{v} = (-1,0,0)$

19b- Did you learn about this in a previous course? Yes No

19c- Was Q19 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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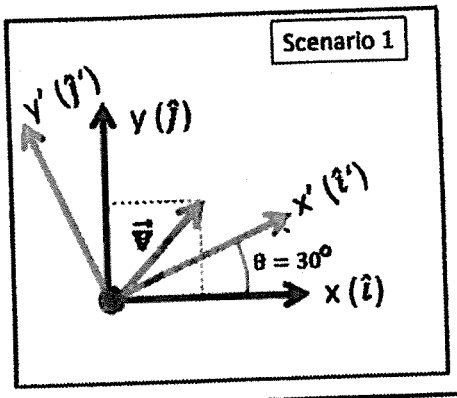
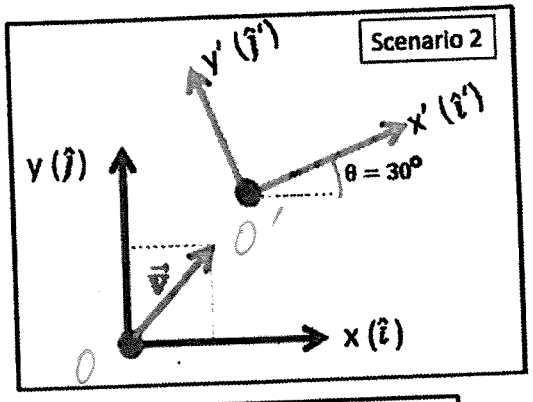
20- We have two scenarios where we want to compute the components of the velocity vector $\vec{v}$ in a rotated coordinate system (i.e. green *primed* frame, rotated by 30 degrees). In scenario 1, the origins of both frames happen to coincide whereas in scenario 2, they do not. Will scenarios 1 and 2 have different coordinate transformation matrices $\vec{M}$, where $\vec{v}\,' = \vec{M}\vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\vec{v}$? (for full marks, explain your answer)





(a) = Yes
(b) = No
Explain: $\vec{v} = \frac{d\vec{x}}{dt} = \frac{d(\vec{x} + \vec{OO'})}{dt}$

20b- Did you learn about this in a previous course? Yes No

20c- Was Q20 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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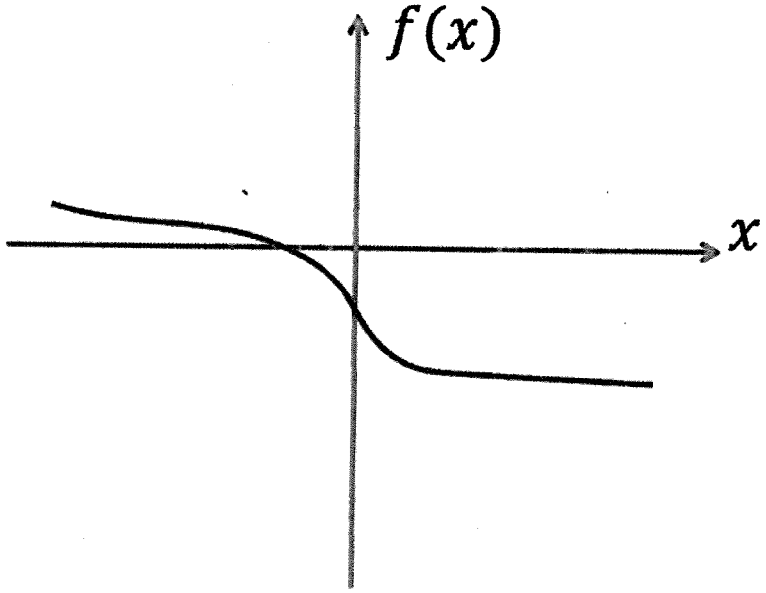
21- The function $f(x)$, shown below, is Taylor expanded to second order in x about $x = 0$:

$f(x) \approx a_0 + a_1 x + a_2 x^2$. Which set of expressions best describes the values of a_0, a_1, and a_2?

$f(0) = a_0 < 0$

$f'(0) = a_1 < 0$

$f''(0) = 2a_2 \longrightarrow$ (c) and (b) are both acceptable.



(a) $a_0 > 0, a_1 > 0, a_2 > 0$

(b) $a_0 < 0, a_1 < 0, a_2 < 0$

(c) $a_0 > 0, a_1 < 0, a_2 < 0$

(d) $a_0 < 0, a_1 > 0, a_2 = 0$

(e) $a_0 < 0, a_1 < 0, a_2 = 0$

21b- Did you learn about this in a previous course? Yes No

21c- Was Q21 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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22- The plot below represents a 2 dimensional cross section of the vector field $\vec{F} = (F_x(x,y), F_y(x,y), 0)$, where the length of the arrow is proportional to the magnitude o the field at that point. What can you say about the divergence of the vector field $(\nabla \cdot \vec{F})$ at the center of the target sign in the lower left corner?

(a) $\nabla \cdot \vec{F} = 0$

(b) $\nabla \cdot \vec{F} > 0$

(c) $\nabla \cdot \vec{F} < 0$

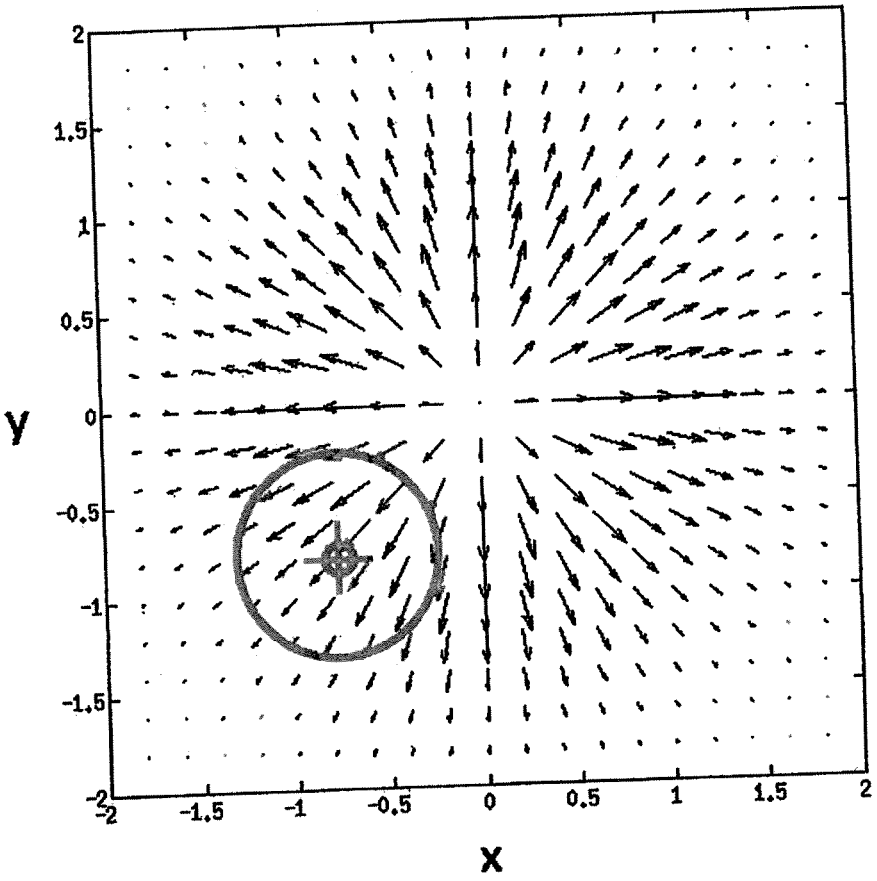
$\vec{F}$ looks like electric field generated by a point charge at origin, $\vec{F} \propto \frac{\hat{r}}{r}$

$\Rightarrow \nabla \cdot \vec{F} \propto \begin{cases} \delta(\vec{r}), & \vec{r}=0 \\ 0, & \text{otherwise.} \end{cases}$

(d) is also acceptable since we actually don't know $\vec{F} \propto \frac{\hat{r}}{r}$

(d) There is not enough information

Please explain your reasoning in choosing your answer:



22b- Did you learn about this in a previous course? Yes No

22c- Was Q22 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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23- The plot below represents a vector field $\vec{B} = \langle B_x(x,y), B_y(x,y), 0\rangle$, where the length of the arrow is proportional to the field magnitude at that point. What can you say about the z-component of $\nabla \times \vec{B}$ at the center of the target sign in the lower left corner? (The positive z direction is toward you, out of the page)

(a) The z-component of $\nabla \times \vec{B}$ is zero

(b) The z-component of $\nabla \times \vec{B}$ is greater than zero

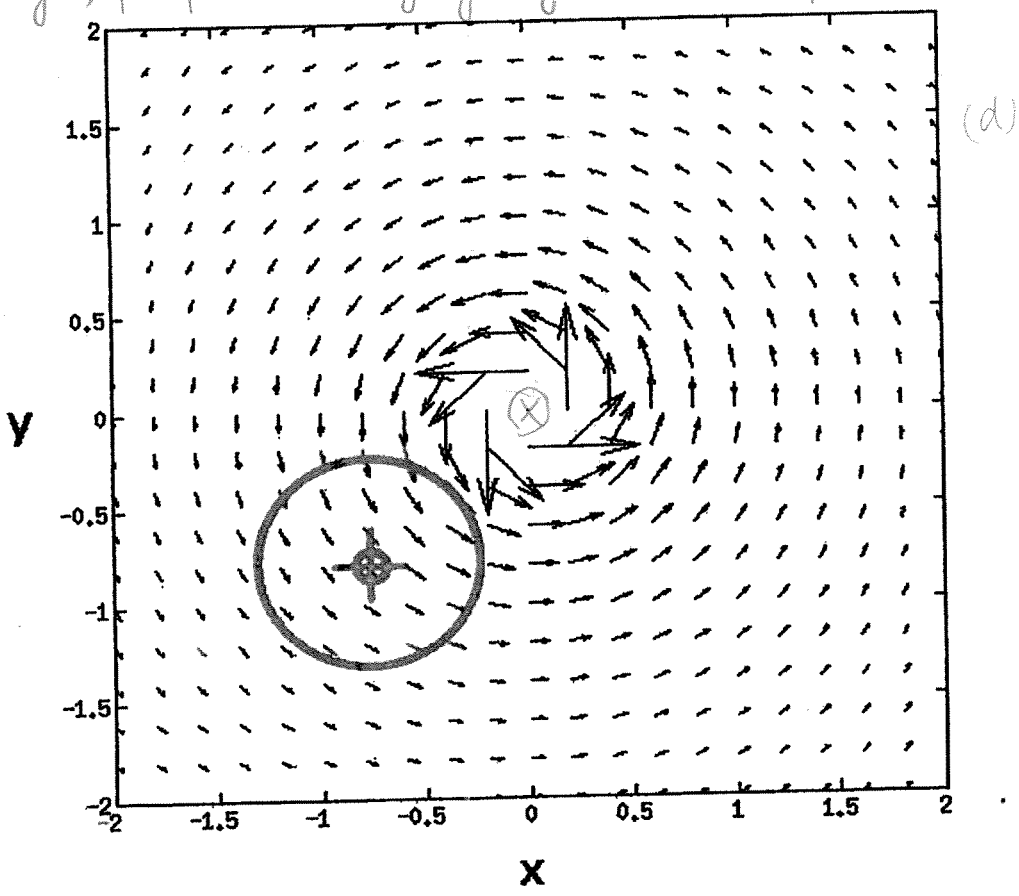
(c) The z-component of $\nabla \times \vec{B}$ is less than zero

(d) There is not enough information

Please explain your reasoning in choosing your answer:

$\vec{B}$ looks like magnetic field generated by a current at the origin, perpendicularly going into the plane. $\nabla \times \vec{B} \propto \begin{cases} \delta(\vec{r}), \vec{r}=0 \\ 0, \text{otherwise} \end{cases}$

(d) is also acceptable.



23b- Please explain your reasoning in choosing your previous answer:

23c- Was Q23 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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24- What is the divergence of the vector field $\vec{A} = xz\,\hat{x} + xy\,\hat{y} + yz\,\hat{z}$?

$$\nabla \cdot \vec{A} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(yz) = z + x + y$$

24b- Was Q24 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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25- What is the curl of the vector field $\vec{B} = y\,\hat{x} - x\,\hat{y} + 5y\,\hat{z}$?

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 5y \end{vmatrix} = 5\hat{x} - 2\hat{z}$$

25b- Was Q25 difficult? 1- Very hard 2- Somewhat hard 3- Avg 4- Somewhat easy 5- Very easy

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How seriously did you just take this diagnostic exam?

(a) I pretty much blew it off, didn't think much about a lot of the answers.
(b) I took it sort of seriously, but when I didn't know an answer I didn't think very hard about it.
(c) I took it seriously, and thought about my answers.

If you imagine getting a letter grade on this test, what do you think that grade would be? ____