Physics 130 Final Exam Solutions

1.) a.) $\hat{a}\cdot\vec{S} = \frac{\hbar}{2}\,\hat{a}\cdot\vec{\sigma} = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{2} & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$

$$\begin{pmatrix} \frac{1}{2} & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{2}{2}\frac{\sqrt{3}}{2} \\ \frac{3}{4} - \frac{1}{4} \end{pmatrix} = 1\cdot\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$$

so $\hat{a}\cdot\vec{S}\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} = +\hbar/2 \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$

in general, all eigenvalues of $\hat{a}\cdot\vec{S}$ are $+\hbar/2$ or $-\hbar/2$

b.) If S^x, S^y, S^z are measured, the possible outcomes are $+\hbar/2$ or $-\hbar/2$. To find the amplitude for the outcome, overlap with the eigenstate

$$a = \langle\uparrow|\psi\rangle \quad \text{or} \quad \langle\downarrow|\psi\rangle$$

The probability is $|a|^2$

For S^z $\quad \langle\uparrow| = (1\ 0) \quad \langle\downarrow| = (0\ 1)$

so

$$a_\uparrow = \frac{\sqrt{3}}{2} \qquad a_\downarrow = \frac{1}{2}$$

$$\text{Prob}(\uparrow) = \frac{3}{4} \qquad \text{Prob}(\downarrow) = \frac{1}{4}$$

c) For S^x $\quad \langle\uparrow| = \frac{1}{\sqrt{2}}(1,1) \qquad \langle\downarrow| = \frac{1}{\sqrt{2}}(1\,-1)$

$$a_\uparrow = \frac{\sqrt{3}+1}{2\sqrt{2}} \qquad a_\downarrow = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$|a_\uparrow|^2 = \frac{3+1+2\sqrt{3}}{8} = \frac{1}{2} + \frac{1}{4}\sqrt{3}$$

$$\text{Prob.}(\uparrow) = \frac{1}{2} + \frac{1}{4}\sqrt{3} \qquad \text{Prob}(\downarrow) = \frac{1}{2} - \frac{1}{4}\sqrt{3}$$

d.) For S^y $\quad \langle\uparrow| = \frac{1}{\sqrt{2}}(1,i)^* = \frac{1}{\sqrt{2}}(1\,-i) \qquad \langle\downarrow| = \frac{1}{\sqrt{2}}(1,i)$

$$a_\uparrow = \frac{\sqrt{3}-i}{2\sqrt{2}} \qquad a_\downarrow = \frac{\sqrt{3}+i}{2\sqrt{2}} \qquad |a|^2 = \frac{3+1}{8} = \frac{1}{2}$$

$$\text{Prob}(\uparrow) = \frac{1}{2} \qquad \text{Prob}(\downarrow) = \frac{1}{2}$$

e.) $$\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \rightarrow e^{-i\frac{\alpha}{2}\sigma^z} \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$$

$$= (\cos 30^\circ - i \sin 30^\circ \sigma^z) \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} (\sqrt{3}/2 - i\,1/2)\,\sqrt{3}/2 \\ (\sqrt{3}/2 + i\,1/2)\,1/2 \end{pmatrix} = \begin{pmatrix} \frac{3-i\sqrt{3}}{4} \\ \frac{\sqrt{3}+i}{4} \end{pmatrix}$$

f) for S^z:

$$a_\uparrow = (1\ 0)\begin{pmatrix}3-i\sqrt{3}\\ \sqrt{3}+i\end{pmatrix}\cdot\frac{1}{4} = \frac{3-i\sqrt{3}}{4} \qquad |a_\uparrow|^2 = \frac{9+3}{16} = \frac{3}{4}$$

$$a_\downarrow = (0\ 1)\begin{pmatrix}3-i\sqrt{3}\\ \sqrt{3}+i\end{pmatrix}\cdot\frac{1}{4} = \frac{\sqrt{3}+i}{4} \qquad |a_\downarrow|^2 = \frac{3+1}{16} = \frac{1}{4}$$

for S^x:

$$a_\uparrow = \frac{1}{\sqrt{2}}(1\ 1)\begin{pmatrix}3-i\sqrt{3}\\ \sqrt{3}+i\end{pmatrix}\frac{1}{4} = \frac{3+\sqrt{3}-i(\sqrt{3}-1)}{4\sqrt{2}}$$

$$|a_\uparrow|^2 = \frac{9+3+6\sqrt{3}+3+1-2\sqrt{3}}{32} = \frac{16+4\sqrt{3}}{32} = \frac{1}{2}+\frac{\sqrt{3}}{8}$$

$$a_\downarrow = \frac{1}{\sqrt{2}}(1\ -1)\begin{pmatrix}3-i\sqrt{3}\\ \sqrt{3}+i\end{pmatrix}\frac{1}{4} = \frac{3-\sqrt{3}-i(\sqrt{3}+1)}{4\sqrt{2}}$$

$$|a_\downarrow|^2 = \frac{9+3-6\sqrt{3}+3+1+2\sqrt{3}}{32} = \frac{1}{2}-\frac{\sqrt{3}}{8}$$

for S^y:

$$a_\uparrow = \frac{1}{\sqrt{2}}(1\ -i)\begin{pmatrix}3-i\sqrt{3}\\ \sqrt{3}+i\end{pmatrix}\frac{1}{4} = \frac{3+1-i\sqrt{3}\cdot 2}{4\sqrt{2}} = \frac{4-2i\sqrt{3}}{4\sqrt{2}}$$

$$|a_\uparrow|^2 = \frac{16+12}{32} = \frac{7}{8}$$

$$a_\downarrow = \frac{1}{\sqrt{2}}(1\ i)\begin{pmatrix}3-i\sqrt{3}\\ \sqrt{3}+i\end{pmatrix}\frac{1}{4} = \frac{3-1-i\sqrt{3}+i\sqrt{3}}{4\sqrt{2}} = \frac{2}{4\sqrt{2}}$$

$$|a_\downarrow|^2 = \frac{1}{8}$$

so S^z $+\frac{1}{2}\hbar$: $3/4$ $\quad -\frac{1}{2}\hbar$: $\frac{1}{4}$

S^x $+\frac{1}{2}\hbar$: $\frac{1}{2}+\frac{\sqrt{3}}{8}$ $\quad -\frac{1}{2}\hbar$: $\frac{1}{2}-\frac{\sqrt{3}}{8}$

S^y $+\frac{1}{2}\hbar$: $7/8$ $\quad -\frac{1}{2}\hbar$: $\frac{1}{8}$

2.) a.)

$$1 = |N|^2 \int_{-\infty}^{\infty} dx \, |e^{-K|x|}|^2$$

$$= 2|N|^2 \int_0^\infty dx \, e^{-2Kx}$$

$$= 2|N|^2 \cdot \frac{1}{2K} = \frac{|N|^2}{K}$$

so $N = \sqrt{K}$

b.) The probability distribution of p is

$$\int \frac{dp}{2\pi\hbar} |\tilde{\psi}(p)|^2$$

where

$$\tilde{\psi}(p) = \int_{-\infty}^{\infty} dx \, e^{-ipx/\hbar} \psi(x)$$

$$= \int_0^\infty dx \, e^{-ipx/\hbar} \sqrt{K} \, e^{-Kx} + \int_{-\infty}^0 dx \, e^{-ipx/\hbar} \sqrt{K} \, e^{Kx}$$

$$= \int_0^\infty dx \, \sqrt{K} \, e^{-(K + iP/\hbar)x} + \int_0^\infty dx \, \sqrt{K} \, e^{-K(x - iP/\hbar)}$$

$$= \frac{\sqrt{K}}{K + iP/\hbar} + \frac{\sqrt{K}}{K - iP/\hbar} = \frac{2K^{3/2}}{|K|^2 + (P/\hbar)^2}$$

Then the probability distribution of p is

$$dp \quad \frac{1}{2\pi\hbar} \; \frac{4\kappa^3}{\left[\kappa^2 + (p/\hbar)^2\right]^2}$$

$$= dp \quad \frac{2\kappa^3}{\pi\hbar} \; \frac{1}{\left[\kappa^2 + (p/\hbar)^2\right]^2}$$

3.) a.) The energy of the state is $E = \frac{1}{2}\hbar\Omega$

so ψ satisfies the Schrödinger equation

$$\frac{1}{2}\hbar\Omega\, \psi(x) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\Omega^2 a^2\right]\psi$$

$$\frac{d^2}{dx^2}\psi(x) = \left[\left(\frac{m^2\Omega^2 a^2}{\hbar^2}\right) - \frac{m\Omega}{\hbar}\right]\psi(x)$$

$$= \left(\frac{a^2}{x_0^4} - \frac{1}{x_0^2}\right)\psi(x)$$

or

$$\frac{d^2}{dx^2}\psi(x) = \frac{1}{x_0^4}(a^2 - x_0^2)\,\psi(x)$$

b.) The regular solution of this equation for $x > a$

is

$$\psi(x) = A\, e^{-(a^2-x_0^2)^{1/2}\frac{x}{x_0^2}}$$

$\psi(x)$ should be continuous at $x=a$. From $x<a$

$$\lim_{x\to a^-}\psi(x) = \frac{1}{(\sqrt{\pi}\,x_0)^{1/2}}\, e^{-\frac{1}{2}\frac{a^2}{x_0^2}}$$

then for $x > a$

$$\psi(x) = \frac{1}{(\sqrt{\pi}\,x_0)^{1/2}}\, e^{-\frac{1}{2}\frac{a^2}{x_0^2}}\, e^{-\left(\frac{a^2}{x_0^2}-1\right)^{1/2}\frac{x-a}{x_0}}$$

c.) The probability distribution of the position is

$$dx\, |\psi(x)|^2$$

for $x > a$

$$|\psi(x)|^2 = \frac{1}{\sqrt{\pi}\, x_0}\, e^{-\frac{a^2}{x_0^2}}\, e^{+2\frac{a}{x_0}\left(\frac{a^2}{x_0^2}-1\right)^{1/2}}\, e^{-2\left(\frac{a^2}{x_0^2}-1\right)^{1/2}\frac{x}{x_0}}$$

$$\int_a^\infty dx\, |\psi(x)|^2 = \frac{1}{\sqrt{\pi}\, x_0}\, e^{-\frac{a^2}{x_0^2}}\, e^{2\frac{a}{x_0}\left(\frac{a^2}{x_0^2}-1\right)^{1/2}}$$

$$\cdot \frac{1}{2\left(\frac{a^2}{x_0^2}-1\right)^{1/2}\frac{1}{x_0}}\, e^{-2\left(\frac{a}{x_0^2}-1\right)^{1/2}\frac{a}{x_0}}$$

$$= \frac{x_0}{\sqrt{\pi}\, x_0}\, e^{-a^2/x_0^2}\, \frac{1}{2\left(a^2/x_0^2-1\right)^{1/2}}$$

$$= \frac{1}{2\sqrt{\pi}}\, \frac{x_0}{\left(a^2-x_0^2\right)^{1/2}}\, e^{-a^2/x_0^2}$$

$$\text{Prob}(x > a) = \frac{1}{2\sqrt{\pi}}\, \frac{x_0}{\left(a^2-x_0^2\right)^{1/2}}\, e^{-a^2/x_0^2}$$

[If you approximated $(a^2-x_0^2)^{1/2} \approx a$ you got full credit.]

4) a) $\langle \psi_f | \vec{\epsilon} \cdot \vec{r} | \psi_i \rangle$

$$= \int d^3x \ \frac{2}{a_0^{3/2}} \frac{1}{\sqrt{4\pi}} e^{-r/a_0} \cdot [x, y, \text{or } z] \ \psi_i(\vec{x})$$

If $i = 2S$ this is of the form

$$\int d^3x \ f(r) \cdot [x, y, \text{or } z] = 0$$

If $i = 2P$ this is of the form

$$\int d^3x \ f(r) \ [x, y, \text{or } z] \cdot g(r) \cdot [x, y, \text{or } z]$$

$= 0$ except for x w. x, y with y, or z with z

so only 3 matrix elements are nonzero

$$\left.\begin{array}{l} \langle 1S | \hat{x} \cdot \vec{r} | 2P_x \rangle \\ \langle 1S | \hat{y} \cdot \vec{r} | 2P_y \rangle \\ \langle 1S | \hat{z} \cdot \vec{r} | 2P_z \rangle \end{array}\right\} \text{ and these are equal.}$$

we need only evaluate

$$\mathcal{M}(2P_x \to 1S + \gamma(\hat{x})) = \langle 1S | \hat{x} \vec{r} | 2P_x \rangle$$

$$= \int d^3x \ \frac{2}{a_0^{3/2}} \frac{1}{\sqrt{4\pi}} e^{-r/a_0} \times \frac{1}{\sqrt{24} \, a_0^{3/2}} \frac{\sqrt{3}}{\sqrt{4\pi}} \frac{x}{a_0} e^{-r/2a_0}$$

$$\langle x \cdot x \rangle = \frac{1}{3} \langle r^2 \rangle$$

$$\int d^3x = 4\pi \int_0^\infty dr \; r^2 \quad \text{for a spherically symmetric integral}$$

$$= \cancel{4\pi} \; \frac{2}{a_0^{3/2}} \frac{1}{\sqrt{24}\, a_0^{3/2}} \frac{\sqrt{3}}{\cancel{4\pi}} \cdot \frac{1}{3} \frac{1}{a_0}$$

$$\cdot \int_0^\infty dr \; r^2 \cdot r^2 \cdot e^{-\frac{3}{2}\frac{r}{a_0}}$$

The value of the integral is

$$\int_0^\infty dr \; r^4 \, e^{-\frac{3}{2}\frac{r}{a_0}} = 4! \left(\frac{2a_0}{3}\right)^5 = 24 \cdot \frac{2^5}{3^5} a_0^5$$

so the result is

$$= \frac{2 \cdot 24 \cdot 32}{\sqrt{24}\; 3^5} \frac{\sqrt{3}}{3} \cdot \frac{1}{a_0^4} a_0^5 \qquad \sqrt{24} = 2 \cdot \sqrt{2} \cdot \sqrt{3}$$

$$= 2 \frac{\sqrt{24}}{\sqrt{3}} \frac{2^5}{3^5} a_0$$

$$\langle 1S | x | 2P_x \rangle = \langle 1S | y | 2P_y \rangle = \langle 1S | z | 2P_z \rangle$$

$$= \frac{2^7}{3^5} \cdot \sqrt{2}\, a_0 =$$ $0.$[redacted] a_0

call this $\equiv \gamma a_0$

b.) Consider $U: \; z \to -z$ $\qquad U^2 = 1$

U commutes with the Hamiltonian of the Hydrogen atom

$$U|1S\rangle = +|1S\rangle \qquad U|2S\rangle = +|2S\rangle$$

$$Uz|2S\rangle = -z|2S\rangle$$

then $\langle 1s|z|2S\rangle = 0$

Proof $\langle 1s|z|2S\rangle = \langle 1s|U^\dagger U\, z|2S\rangle$

$$= (+1)(-1)\langle 1s|z|2S\rangle = -\langle 1s|z|2S\rangle$$

this implies $\langle 1s|z|2S\rangle = 0$

c.) We need the matrix elements of $\Delta H = eEz$ between $2S, 2P_x\ 2P_y\ 2P_z$. Actually, the only nonzero matrix element is

$$\langle 2P_z|\, eEz\,|2S\rangle \quad \text{and the conjugate}$$

$$= \int d^3x \left(\frac{1}{\sqrt{24}\,a_0^{3/2}} \frac{\sqrt{3}}{\sqrt{4\pi}} \frac{z}{a_0} e^{-r/2a_0}\right) eEz \left(\frac{1}{\sqrt{2}\,a_0^{3/2}} \frac{1}{\sqrt{4\pi}} \left(1-\frac{r}{2a_0}\right) e^{-r/2a_0}\right)$$

$$= \frac{1}{\sqrt{24}\sqrt{2}} \frac{\sqrt{3}}{4\pi} \frac{1}{a_0^4}\, eE$$

$$\int_0^\infty dr\, r^2 \left(1 - r/a_0\right) e^{-r/a_0} \cdot \frac{4\pi}{3} r^2$$

$$= \frac{1}{4 \cdot 4\cancel{\pi}} \frac{4\cancel{\pi}}{3} \frac{1}{a_0^4} eE$$

$$\cdot (4! - 5!)\, a_0^5$$

$$= \frac{1}{4 \cdot 3} \cdot 24 \cdot (-4)\, a_0 e E$$

$$= -8 a_0 e E$$

so

$$\Delta H = \begin{pmatrix} 0 & -8eEa_0 \\ -8eEa_0 & 0 \end{pmatrix} \begin{array}{l} |2S\rangle \\ |2P_z\rangle \end{array}$$

the energies and eigenstates are:

$$E = -\frac{R_y}{4} + 8eE_0 a_0 \qquad \frac{1}{\sqrt{2}}\left(|1S\rangle - |2P_z\rangle\right) = |-\rangle$$

$$E = -\frac{R_y}{4} - 8eE_0 a_0 \qquad \frac{1}{\sqrt{2}}\left(|2S\rangle + |2P_z\rangle\right) = |+\rangle$$

d.)

$$\frac{\text{amplitude}\left(|+\rangle \to |1S\rangle + |\gamma_z\rangle\right)}{\text{amplitude}\left(|2P_x \to 1S + \gamma_x\right)} = \frac{1}{\sqrt{2}}$$

so

$$\frac{\text{rate}\left(|+\rangle \to 1S + \gamma_z\right)}{\text{rate}\left(2P_x \to 1S + \gamma_x\right)} = \frac{1}{2}$$

similarly

$$\frac{\text{rate}\left(|-\rangle \to 1S + \gamma_z\right)}{\text{rate}\left(2P_x \to 1S + \gamma_x\right)} = \frac{1}{2}$$

5.) a.) The Hamiltonian is

$$-\Delta \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{acting on} \quad \begin{matrix} |1\rangle \\ |2\rangle \\ |3\rangle \\ |4\rangle \end{matrix}$$

b.)
The eigenvectors are (by guessing)

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad E = -3\Delta$$

$$\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad E = +\Delta$$

$$\begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix} \qquad E = +\Delta$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \qquad E = +\Delta$$

In fact, any vector orthogonal to $(1\,1\,1\,1)$ has eigenvalue $E = +\Delta$. To see this rewrite

$$H = \Delta \cdot \underline{1} - \Delta \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

c.)

$$\sum E_i = -3\Delta + \Delta + \Delta + \Delta = 0$$

this is required, since

$$\text{trace}(H) = 0$$

d) $|4\rangle$ is the state $\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$. we can write this in terms of eigenvectors as

$$|4(t)\rangle\Big|_{t=0} = \frac{1}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}-1\\-1\\-1\\3\end{pmatrix}$$

Giving the pieces their proper time-dependence.

$$|4(t)\rangle = \frac{1}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} e^{+i\,3\Delta t/\hbar} + \frac{1}{4}\begin{pmatrix}-1\\-1\\-1\\3\end{pmatrix} e^{-i\frac{\Delta t}{\hbar}}$$

e.) the two pieces come back into phase when

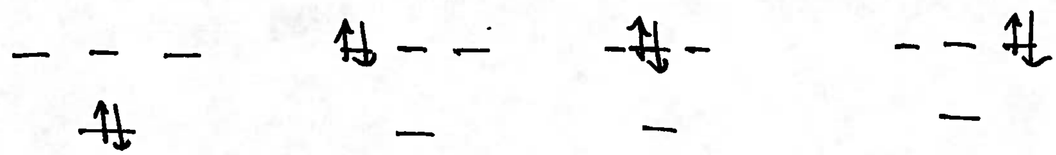
$$e^{i\,4\Delta t/\hbar} = 1$$

or $\quad \dfrac{4\Delta t}{\hbar} = 2\pi \qquad t = \dfrac{\pi}{2\Delta}\hbar$

f.) There are 4 state × 2 spins = 8 electron states, to be populated by 2 electrons in antisymmetric combinations. The total number of states is

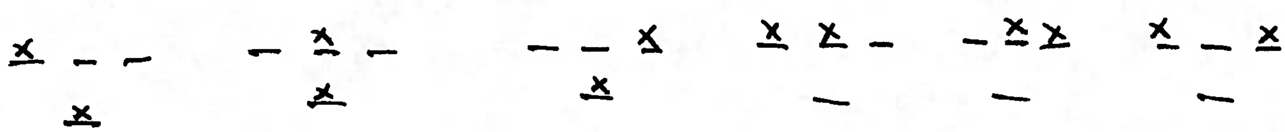
$$\frac{8\cdot 7}{2} = 28$$

This includes 4 states with both electrons in the same state, necessarily with total spin 0



$E = -6\Delta$	$E = 2\Delta$	$E = 2\Delta$	$E = 2\Delta$
$S = 0$	$S = 0$	$S = 0$	$S = 0$

There are $\frac{4\cdot 3}{2} = 6$ ways to put the electrons in 2 different states. Here both spin 0 (1 state) and spin 1 (3 states) are possible, for a total of 24 states.



$E = -2\Delta$	$E = -2\Delta$	$E = -2\Delta$	$E = 2\Delta$	$E = 2\Delta$	$E = 2\Delta$
$S = 0, 1$	$S = 0,1$	$S = 0,1$	$S = 0,1$	$S = 0,1$	$S = 0,1$