February 25

Angular Momentum

In our discussion of the basic principles of quantum mechanics, we introduced the unitary transformations that generate time translations, space translations, and rotations. From these, we defined the Hermitian operators

$$H \quad \vec{P} \quad \vec{J}$$

associated with the fundamental conservation laws of energy, momentum, and angular momentum. We studied H and $\vec{P}$ in some detail in the past few lectures, but I postponed the discussion of $\vec{J}$. We are now ready to analyze $\vec{J}$ and understand its properties.

As a preface, let's review what we learned about $\vec{P}$. By virtue of the role of $\vec{P}$ as the generator of translations, this operator satisfies the commutation relation

$$[X, P] = i\hbar$$

or, in 3 dimensions

$$[X^i, P^j] = i\hbar\,\delta^{ij}$$

From this, we found the representation of the operator $\vec{P}$ on Schrödinger wavefunctions

$$P^j = -i\hbar \frac{\partial}{\partial x^j} \qquad \text{or} \qquad \vec{P} = -i\hbar \vec{\nabla}$$

In this lecture, I will use a similar logic to construct the commutation relation of $\vec{J}$ and the explicit form of this operator as it acts on a wavefunction.

We can start from a part of the theory that is already familiar to you, the description of rotations on vectors. We represent a vector in 3 dimensions as a triplet of real numbers

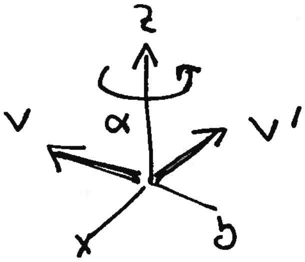
$$\vec{V} = (V^x, V^y, V^z)$$

A rotation by an angle α about the $\hat{z}$ axis transforms this set of numbers by the matrix action

$$\begin{pmatrix} V^x \\ V^y \\ V^z \end{pmatrix} \to \begin{pmatrix} V^{x'} \\ V^{y'} \\ V^{z'} \end{pmatrix} = R(\alpha) \begin{pmatrix} V^x \\ V^y \\ V^z \end{pmatrix}$$

where $R(\alpha)$, the rotation matrix, has the explicit form

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Rotations are infinitesimally generated. Thus, we can understand the form of a rotation matrix by considering it as a sequence of rotations through a very small angle. Let $\alpha = N\epsilon$. Then

$$R(\alpha) = \left[R(\epsilon)\right]^N$$

Expanding the rotation matrix given above,

$$R(\epsilon) = 1 + \epsilon \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots$$

The matrix $R(\alpha)$ is norm preserving in a real-valued vector space, so $R(\alpha)$ is a unitary matrix. We can then compare the infinitesimal form of the rotation matrix to our standard form for an infinitesimal unitary matrix

$$U(\varepsilon) = 1 - i\epsilon G + \cdots$$

where G is the generator, a Hermitian operator. In this case, the generator G is the matrix

$$S^z = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is an antisymmetric Hermitian matrix.

According to our general theory, we can reconstruct the finite rotation matrix by taking the exponential of the generator,

$$R(\alpha) = e^{-i\alpha S^z}$$

You can check explicitly that this works. The power series of the exponential is

$$R(\alpha) = 1 - i\alpha S^z + \frac{(-i\alpha)^2}{2!}(S^z)^2 + \frac{(-i\alpha)^3}{3!}(S^z)^3 + \cdots$$

Now

$$(S^z)^2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \qquad (S^z)^3 = S^z$$

Using these identities, it is not difficult to write each term in the power series as a 3×3 matrix and show that the power series for the matrix elements sum up to the $\sin\alpha$ and $\cos\alpha$ factors above in the correct way.

I would now like to ask, what is the most general rotation? I will define a rotation as a linear transformation on 3-dimensional, real-valued vectors that preserves the vector product

$$\vec{v}\cdot\vec{w} = (R\vec{v})\cdot(R\vec{w})$$

and is infinitesimally generated. Actually, there is a transformation called *parity* that satisfies all of the criteria except that it cannot be obtained from the identity as a product of infinitesimal rotations,

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The set of all infinitesimally generated rotations is called $SO(3)$; the set of tranformations that includes these, the parity operator, and products of the two is called $O(3)$.

If R preserves the vector product, then

$$v^i \omega^i = (R^{ij} v^j)(R^{ik} w^k) = v^j (R^{ij} R^{ik}) w^k$$

or

$$R^T R = 1$$

The form of this equation for an infinitesimal operation is

$$(1 - i\epsilon G^T + \cdots)(1 - i\epsilon G + \cdots) = 1$$

which implies

$$G^T + G = 0$$

Thus, in general, the generator of a rotation is an antisymmetric matrix. Since it must also be Hermitian, it must be imaginary-valued.

In 3 dimensions, there are precisely 3 independent antisymmetric 3 × 3 matrices. These are

$$S^x = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S^y = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad S^z = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These matrices can be conveniently written as

$$(S^k)_{ij} = i\,\epsilon_{ikj} \qquad \begin{array}{l} \epsilon_{123} = +1 \\ \epsilon_{213} = -1 \quad \text{etc.} \end{array}$$

The infinitesimal rotation of a vector is then

$$(R(\varepsilon)V)^i = (\delta^{ij} - i \cdot i\ \epsilon_{ikj}\, \epsilon^k + \cdots)\, V^j$$

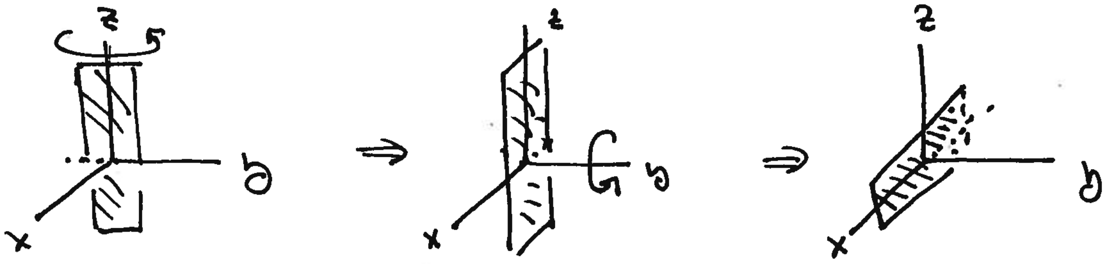
or

$$R(\varepsilon)\vec{V} = \vec{V} + \vec{\epsilon} \times \vec{V} + \cdots$$

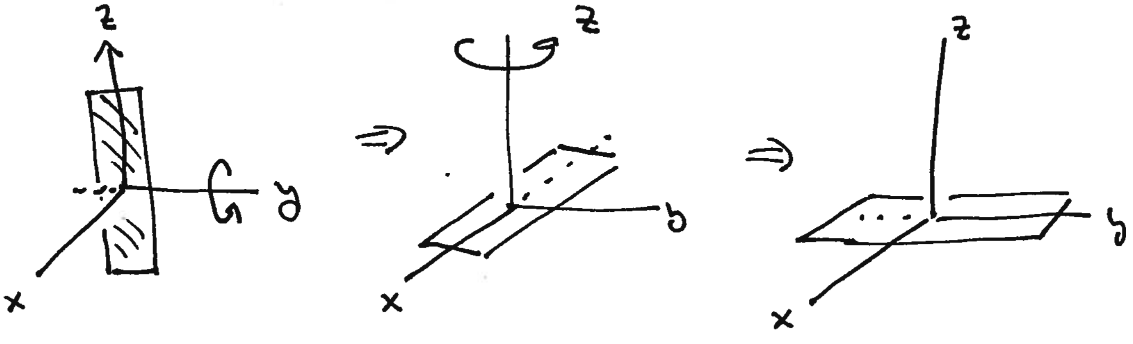
This equation describes the rotation of a vector by an angle $|\vec{\epsilon}|$ about an axis $\hat{\epsilon}$ (the unit vector in the direction of $\vec{\epsilon}$). This argument shows that every rotation in the sense defined above is in fact a rotation about a definite axis. We can then parametrize a general rotation in 3-dimensional space as a vector $\vec{\alpha}$, where $\hat{\alpha}$ is the axis of the rotation and $|\vec{\alpha}|$ is the angle of the rotation. The matrix corresponding to this finite rotation can be written

$$R(\vec{\alpha}) = e^{-i\vec{\alpha}\cdot\vec{S}}$$

Notice that the matrices S^k do not commute with one another. This is expected, because two general rotations in 3 dimensions do not commute. For example, a rotation by 90° about $\hat{z}$ followed by a rotation by 90° about $\hat{y}$



does not give the same result as a rotation by 90° about $\hat{y}$ followed by a rotation by 90° about $\hat{z}$.



The non-commutation properties of the S^k actually give us a relatively straightforward way to understand the non-commutativity of more general rotations. The commutation relations of the S^k are actually quite simple. For example,

$$[S^x, S^y] = (i)^2 \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right]$$

$$= (i)^2 \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$= (i)^2 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = i S^z$$

More generally, as you can easily show,

$$[S^i, S^j] = i \epsilon_{ijk} S^k$$

It can be shown that this commutation relation determines the multiplication law for any two 3-dimensional rotations. Write

$$R(\vec{\alpha}) = e^{-i \vec{\alpha} \cdot \vec{S}} \qquad R(\vec{\beta}) = e^{-i \vec{\beta} \cdot \vec{S}}$$

Then the product of the two rotations is the result of the matrix multiplication

$$R(\vec{\alpha}) R(\vec{\beta}) = e^{-i \vec{\gamma} \cdot \vec{S}}$$

We can find the result by expressing each exponential as a power series, multiplying the two series together, and then forming the result into the series of an exponential. The only ambiguity in carrying out this calculation is that the generators S^k might appear in a string, e.g.,

$$S^x S^y S^y S^z S^x$$

that needs to be re-ordered. The commutation relation of the S^k tells us how to put this string into a canonical order.

Let's now turn to the action of rotations on Schrödinger wavefunctions. To begin this discussion, I would like to discuss the action of translations on Schrödinger wavefuntions. The translation

$$\vec{x} \longrightarrow \vec{x}' = \vec{x} + \vec{a}$$

transforms a wavefunction $\psi(\vec{x})$ according to

$$\psi(\vec{x}) \longrightarrow \psi'(\vec{x}) = \psi(\vec{x} - \vec{a})$$

To justify this, note that, if $\psi(\vec{x})$ has a maximum at $\vec{x} = \vec{x}_0$, the maximum of $\psi'(\vec{x})$ will occur at $\vec{x} = \vec{x}_0 + \vec{a}$. For an infinitesimal translation, $|\vec{a}|$ very small,

$$\psi(\vec{x}) \rightarrow \psi'(\vec{x}) = \psi(\vec{x}) - \vec{a} \cdot \vec{\nabla} \psi(\vec{x}) + \cdots$$

In our description of the Hilbert space of Schrödinger wavefunctions, the translation of a wavefunction was given by

$$\psi \rightarrow U(\vec{a}) \psi = \left(1 - i \frac{\vec{a} \cdot \vec{P}}{\hbar}\right) \psi$$

Identifying these expressions, we see that the operator $\vec{P}$ acts on Schrödinger wavefunctions as

$$\vec{P} = -i\hbar \vec{\nabla}$$

in accord with the representation that we derived earlier.

For rotations, we can follow this same line of argument. A rotation acts on a Schrödinger wavefunction as

$$\psi(\vec{x}) \rightarrow \psi'(\vec{x}) = \psi(R^{-1}(\vec{\alpha})\,\vec{x})$$

The infinitesimal form of the rotation is

$$R^{-1}(\alpha)\,\vec{x} = \vec{x} - \vec{\alpha} \times \vec{x} + \cdots$$

so

$$\begin{aligned} \psi'(\vec{x}) &= \psi(\vec{x}) - (\vec{\alpha} \times \vec{x}) \cdot \vec{\nabla}\,\psi(\vec{x}) + \cdots \\ &= \psi(\vec{x}) - \vec{\alpha} \cdot (\vec{x} \times \vec{\nabla})\,\psi(\vec{x}) + \cdots \end{aligned}$$

This expression must be identified with

$$U(\vec{\alpha})\,\psi = \left(1 - i\,\frac{\vec{\alpha} \cdot \vec{J}}{\hbar} + \cdots\right)\psi$$

Then we find

$$\vec{J} = -i\hbar\;\vec{x} \times \vec{\nabla}$$

or, in components,

$$J^j = -i\hbar\, \epsilon^{jkl}\, x^k \frac{\partial}{\partial x^l}$$

I apologize that, at this point, I am going to change the notation slightly. Conventionally, we reserve $\vec{J}$ to represent the total angular moment of a system. When we discussed real atoms, we saw that an electron actually carries angular momentum in two places, first, in the angular momentum of its orbital motion, second, in the mysterious spin degree of freedom. Then, for an electron, we write

$$\vec{J} = \vec{L} + \vec{S}$$

where $\vec{L}$ represents the orbital part of the angular momentu and $\vec{S}$ represents the spin angular momentum. So, from here on, I will define

$$L^j = -i\hbar\, \epsilon^{jkl}\, x^k \frac{\partial}{\partial x^l}$$

and call this orbital angular momentum. For a simple particles described completely by its Schrödinger wavefunction, this is the only source of angular momentum.

Notice that

$$L^j = \epsilon^{jkl}\, x^k P^l \quad \text{or} \quad \vec{L} = \vec{X} \times \vec{P}$$

This dovetails nicely into your knowledge of classical particle mechanics.

As with $\vec{P}$, it is important to work out the commutation relation of $\vec{L}$ with $\vec{X}$. This is readily done:

$$
\begin{aligned}
[X^i, L^j] &= \epsilon^{jkl}\,[X^i, X^k P^l] \\
&= \epsilon^{jkl}\,X^k\,[X^i, P^l] \\
&= \epsilon^{jkl}\,X^k\,i\hbar\,\delta^{il} = i\hbar\,\epsilon^{ijk}\,X^k
\end{aligned}
$$

Similarly, we can work out the commutation relation of $\vec{L}$ with $\vec{P}$,

$$
\begin{aligned}
[P^i, L^j] &= \epsilon^{jkl}\,[P^i, X^k P^l] \\
&= \epsilon^{jkl}\,[P^i, X^k]\,P^l \\
&= \epsilon^{jkl}\,-i\hbar\,\delta^{ik}\,P^l = -i\hbar\,\epsilon^{jil}\,P^l = i\hbar\,\epsilon^{ijl}P^l
\end{aligned}
$$

Again,

$$
[X^i, L^j] = i\hbar\,\epsilon^{ijk}X^k \qquad\qquad [P^i, L^j] = i\hbar\,\epsilon^{ijk}P^k
$$

We can rewrite the commutator of $\vec{X}$ with $\vec{L}$ in the following suggestive way:

$$
\begin{aligned}
X^i\left(1 - i\frac{\vec{\alpha}\cdot\vec{L}}{\hbar}\right) &= \left(1 - i\frac{\vec{\alpha}\cdot\vec{L}}{\hbar}\right)X^i + \epsilon^{ijk}\alpha^j X^k \\
&= \left(1 - i\frac{\vec{\alpha}\cdot\vec{L}}{\hbar}\right)\left(X^i + (\vec{\alpha}\times\vec{X})^i + \mathcal{O}(\alpha^2)\right)
\end{aligned}
$$

Since

$$
\mathcal{R}(\vec{\alpha})\vec{X} = \vec{X} + \vec{\alpha}\times\vec{X} + \dots
$$

this is the infinitesimal form of the relation

$$\vec{X}\, U(\vec{\alpha}) = U(\vec{\alpha})\, (R(\vec{\alpha})\vec{X})$$

This equation says that if we first measure $\vec{X}$, then rotate the answer, and then rotate the system, we get the same result as we do when we first rotate the system and then measure $\vec{X}$. A similar result applies for $\vec{P}$,

$$\vec{P}\, U(\vec{\alpha}) = U(\vec{\alpha})\, (R(\vec{\alpha})\vec{P})$$

In fact, if $\vec{A}$ is any operator that transforms as a vector under 3-dimensional rotations (for example, the dipole moment of a molecule), it should obey the same relation. This relation is encoded in the commutation relation

$$[A^i, L^j] = i\hbar\epsilon^{ijk} A^k$$

Actually, angular momentum is itself a vector operator in 3 dimensions. Thus, we expect

$$[L^i, L^j] = i\,\epsilon^{ijk} L^k$$

This can be proved directly. First,

$$\begin{aligned}
[L^i, L^j] &= \epsilon^{iab}\,[X^a P^b, L^j] \\
&= \epsilon^{iab}\left([X^a, L^j]\,P^b + X^a\,[P^b, L^j]\right) \\
&= \epsilon^{iab}\left[(i\hbar\,\epsilon^{ajc} X^c)P^b + X^a(i\hbar\,\epsilon^{bjd} P^d)\right] \\
&= i\hbar\,\epsilon^{iab}\epsilon^{ajc}\, X^c P^b + i\hbar\,\epsilon^{iab}\epsilon^{bjd}\, X^a P^d
\end{aligned}$$

The ϵ^{ijk} symbol obeys the identity

$$\epsilon^{abc}\epsilon^{ade} = \delta^{bd}\delta^{ce} - \delta^{be}\delta^{cd}$$

Applying this identity, the above expression becomes

$$= i\hbar\,(\delta^{bj}\delta^{ic} - \delta^{bc}\delta^{ij})\,X^c P^b + i\hbar\,(\delta^{ij}\delta^{ad} - \delta^{id}\delta^{aj})\,X^a P^d$$

$$= i\hbar\,[X^i P^j - \delta^{ij}\,\vec{X}\cdot\vec{P} + \delta^{ij}\,\vec{X}\cdot\vec{P} - X^j P^i]$$

$$= i\hbar\,[X^i P^j - X^j P^i]$$

and, applying the identity in reverse, we find

$$= i\hbar\,\epsilon^{ijk}\epsilon^{kab}\,X^a P^b = i\hbar\,\epsilon^{ijk}L^k$$

The commutation relation of L^i with L^j tells us two things: First, it tells us that $\vec{L}$ transforms $\vec{L}$ as a vector under 3-dimensional rotations, second, it tells us that the components of $\vec{L}$ have the proper commutation relations with one another so that their exponentials

$$U(\vec{\alpha}) = e^{-i\vec{\alpha}\cdot\vec{L}/\hbar}$$

have the standard multiplication law of 3-dimensional rotations.

A scalar quantity should be invariant to rotations and therefore should commute with $\vec{L}$. Let $\vec{A}$ be a vector operator with the commutation relation with $\vec{L}$ given above. Then

$$[A^2, L^j] = [A^i A^i, L^j] = [A^i, L^j] A^i + A^i [A^i, L^j]$$
$$= i\hbar\, \epsilon^{ijk} A^k A^i + A^i \cdot i\hbar\, \epsilon^{ijk} A^k$$
$$= i\hbar\, \epsilon^{ijk} (A^k A^i + A^i A^k)$$

The contraction of a symmetric and an antisymmetric matrix is zero; thus,

$$[A^2, L^j] = 0$$

Note that, in this derivation, we did not need to assume that the components of $\vec{A}$ commute with one another. In particular, this argument is valid for the square of $\vec{L}$ itself and implies

$$[L^2, L^j] = 0$$

To complete our study of the action of $\vec{L}$ on Schrödinger wavefunctions, let's work out the form of $\vec{L}$ in polar coordinates. Begin with

$$L^z = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

Using

$$r = (x^2 + y^2 + z^2)^{1/2} \qquad \cos\theta = \frac{z}{r} \qquad \phi = \tan^{-1} \frac{y}{x}$$

and the transformation of partial derivatives

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \cos\theta}{\partial x}\frac{\partial}{\partial \cos\theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

we see that

$$\frac{\partial}{\partial x} = \frac{x}{r}\frac{\partial}{\partial r} - \frac{xz}{r^3}\frac{\partial}{\partial \cos\theta} - \frac{y}{x^2+y^2}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \frac{y}{r}\frac{\partial}{\partial r} - \frac{yz}{r^3}\frac{\partial}{\partial \cos\theta} + \frac{x}{x^2+y^2}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = \frac{z}{r}\frac{\partial}{\partial r} + \frac{x^2+y^2}{r^3}\frac{\partial}{\partial \cos\theta}$$

Then

$$L^z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$= -i\hbar\left[\left(\frac{xy}{r} - \frac{yx}{r}\right)\frac{\partial}{\partial r} + \left(-\frac{xyz}{r^3} + \frac{yxz}{r^3}\right)\frac{\partial}{\partial \cos\theta} + \left(\frac{x^2+y^2}{x^2+y^2}\right)\frac{\partial}{\partial \phi}\right]$$

This expression collapses to

$$L^z = -i\hbar\frac{\partial}{\partial \phi}$$

The eigenfunctions of this operator are

$$e^{im\phi} \qquad m = \text{integer}$$

and the corresponding eigenvalues are

$$L^z = \hbar m$$

This is the association that was suggested by the form of the Schrödinger equation for a particle in a spherically symmetric potential. Now we have a first-principles derivation.

The expressions for L^x and L^y are somewhat more complicated. For L^x,

$$L^x = -i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right)$$

$$= -i\hbar \left[\left(\frac{y(x^2+y^2)}{r^3} + \frac{yz^2}{r^3} \right) \frac{\partial}{\partial \cos\theta} - \frac{xz}{x^2+y^2} \frac{\partial}{\partial \phi} \right]$$

$$= -i\hbar \left[\frac{y r^2}{r^3} \frac{\partial}{\partial \cos\theta} - \frac{xz}{x^2+y^2} \frac{\partial}{\partial \phi} \right]$$

$$= -i\hbar \left[-\sin\phi \frac{\partial}{\partial \theta} - \frac{\cos\theta}{\sin\theta} \cos\phi \frac{\partial}{\partial \phi} \right]$$

For L^y,

$$L^y = -i\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right)$$

$$= -i\hbar \left[-\left(\frac{xz^2}{r^3} + \frac{x(x^2+y^2)}{r^3} \right) \frac{\partial}{\partial \cos\theta} - \frac{yz}{x^2+y^2} \frac{\partial}{\partial \phi} \right]$$

$$= -i\hbar \left[\cos\phi \frac{\partial}{\partial \theta} - \frac{\cos\theta}{\sin\theta} \sin\phi \frac{\partial}{\partial \phi} \right]$$

Now we can form $L^2 = (L^x)^2 + (L^y)^2 + (L^z)^2$,

$$L^2 = -\hbar^2 \Big[\sin^2\phi \frac{\partial^2}{\partial \theta^2} + \sin\phi \frac{\partial}{\partial \theta} \left(\frac{\cos\theta}{\sin\theta} \right) \cos\phi \frac{\partial}{\partial \phi}$$

$$+ \frac{\cos\theta}{\sin\theta} \cos\phi \frac{\partial}{\partial \phi} \sin\phi \frac{\partial}{\partial \theta} + \left(\frac{\cos\theta}{\sin\theta} \right)^2 \cos\phi \frac{\partial}{\partial \phi} \cos\phi \frac{\partial}{\partial \phi}$$

$$+ \cos^2\phi \frac{\partial^2}{\partial \theta^2}$$ [redacted] $$\cos\phi \frac{\partial}{\partial \theta} \left(\frac{\cos\theta}{\sin\theta} \right) \sin\phi \frac{\partial}{\partial \phi}$$

$$- \frac{\cos\theta}{\sin\theta} \sin\phi \frac{\partial}{\partial\phi} \cos\phi \frac{\partial}{\partial\theta} + \left(\frac{\cos\theta}{\sin\theta}\right)^2 \sin\phi \frac{\partial}{\partial\phi} \sin\phi \frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi^2} \Big]$$

$$= -\hbar^2 \Big[\frac{\partial^2}{\partial\theta^2} + 2\sin\phi\cos\phi \left(\frac{\cos\theta}{\sin\theta} \left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\phi}\right) + \frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{\sin\theta}\right) \frac{\partial}{\partial\phi} \right)$$

$$+ \frac{\cos\theta}{\sin\theta} \cos^2\phi \frac{\partial}{\partial\theta} + \left(\frac{\cos\theta}{\sin\theta}\right)^2 (-\cos\phi\sin\phi) \frac{\partial}{\partial\phi} + \left(\frac{\cos\theta}{\sin\theta}\cos\phi\right)^2 \frac{\partial^2}{\partial\phi^2}$$

$$- 2\sin\phi\cos\phi \left(\frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\theta} \frac{\cos\theta}{\sin\theta} \right) \frac{\partial}{\partial\phi}$$

$$+ \frac{\cos\theta}{\sin\theta} \sin^2\phi \frac{\partial}{\partial\theta} + \left(\frac{\cos\theta}{\sin\theta}\right)^2 \sin\phi\cos\phi \frac{\partial}{\partial\phi} + \left(\frac{\cos\theta}{\sin\theta}\sin\phi\right)^2 \frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\phi^2}$$

Many terms cancel, but some are left, including terms created by the action of $\partial/\partial\theta$ and $\partial/\partial\phi$ on θ- and ϕ-dependent coefficients

$$= -\hbar^2 \Big[\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta} (\cos^2\phi + \sin^2\phi) \frac{\partial}{\partial\theta}$$

$$+ \left[\frac{\cos^2\theta}{\sin^2\theta} (\cos^2\phi + \sin^2\phi) + 1 \right] \frac{\partial^2}{\partial\phi^2} \Big]$$

The final result is

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

This is exactly the operator whose eigenfunctions are the spherical harmonics. This completes the full, first-principles, derivation of the equation

$$-\frac{\hbar^2}{2m} \nabla^2 = -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{L^2}{2mr^2}$$

with the identification of the operator L^2 with angular momentum.

You should recall that the eigenvalues of the operator L^2 are

$$L^2 = \hbar^2 \, l(l+1)$$

for $\ell = 0, 1, 2, \ldots$. The maximum value of $(L^z)^2$ in a state with $\ell = \ell$ is

$$(L^z)^2 = \hbar^2 \ell^2$$

We must have

$$L^2 \geq (L^z)^2$$

simply because $(L^x)^2 + (L^y)^2$ is a sum of squares. However, it is somewhat surprising that equality is not allowed. This is true because

$$[L^x, L^z] \neq 0 \qquad [L^y, L^z] \neq 0$$

If L^z is known precisely, the values of L^x, L^y must be indefinite. Then the expectation values of $(L^x)^2$ and $(L^y)^2$ must be strictly positive.