The Four-Dimensional Formalism of Special Relativity

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The purpose of this handout is to summarize the four-dimensional formalism of Special Relativity. All of the usual effects – time dilation, length contraction, the relativity of simultaneity, and so on – can be described as the invariance of the laws of physics under changes of coordinate systems in four dimensional spacetime which leave all geometric properties such as lengths and angles invariant.

Prerequisites: This handout picks up where the three-dimensional formalism left off; it assumes a familiarity with the ideas of time dilation, length contraction, and the relativity of simultaneity. These ideas are covered in detail in Taylor and Wheeler [1] and Mermin. [2] It assumes familiarity with basic matrix algebra.

I. PRÉLUDE 1: RELATIVITY AND COORDINATE TRANSFORMATIONS

In the previous lecture, we began with three postulates:

1. The laws of physics are the same in all inertial reference frames; that is, any two observers who differ only by their relative (constant) velocity see the same laws of physics.
2. The speed of light is a constant (c) in all inertial reference frames.
3. An *event* is defined to be the coincidence of two things at the same point in space and time. All observers can agree on events. For example, if a clock is rigged to ring a bell if it is hit by a particle just as it strikes twelve, everyone will agree on whether or not it rang the bell.

The first and third of these are fairly obvious statements; the second, which consists of taking Michelson and Morley's experimental result at face value, is highly nontrivial. If Alice sees Bob pass her at velocity v, and Bob shines a flashlight ahead of him, Alice naïvely ought to see the light recede from her at velocity $v + c$. For the second postulate to hold, therefore, we must carefully reconsider our interpretation of the meaning of length and time.

A careful analysis of various situations involving observers carrying rods and emitting light pulses while in motion led us to the conclusion that, if these three postulates are to be true, the notion of elapsed time and distance are not universal, but rather depend on the observer; for example, a person approaching a rod lengthwise at speed v will see its length contracted by a factor $\sqrt{1 - v^2/c^2}$. Several such rules were derived, some of them somewhat complicated, so it is useful to summarize them all in the language of coordinate transformations.

We can construct a coordinate system out of components found around a sufficiently large house. The obvious component is a single infinite rod, representing the x axis, with marks placed at regular intervals. At each point on this rod we attach perpendicular rods representing the y and z axes, and at each coordinate point we also place a clock which will measure elapsed time, our t axis. Coordinates are therefore described by four numbers, and by convention time is taken to be the zeroth (rather than the fourth) coordinate.[1] These will be denoted by four-component vectors. $x = (t, x, y, z)$.

Consider now taking two identical coordinate systems, x and x', and setting x' in motion with speed v along the x axis. We would like to express the coordinates of a point in space and time in the x' system in terms of the x system.

This is not very complicated. Start with the x coordinate. Since the primed system is in motion, its hash marks appear to be contracted by a factor $\sqrt{1 - v^2/c^2}$; therefore it takes more hash marks to reach a given point, so $x' \sim x/\sqrt{1 - v^2/c^2}$. Adding in the fact that this coordinate system is also moving, we find

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \ . \tag{1}$$

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[1] This is because the universe programs in C.

y' and z' are even simpler; since coordinates perpendicular to the direction of motion are not affected by relativity,

$$y' = y; \;\; z' = z \; . \tag{2}$$

Finally, the shift in time follows from our simultaneity formula. Say we synchronize the clocks at some instant so that both of the clocks at $x = 0$ (the ones in each coordinate system) read zero at once; then at that instant, the clock at position x belonging to the moving coordinate system reads $-(xv/c^2)/\sqrt{1-v^2/c^2}$. (At this instant all of the clocks in the stationary coordinate system read zero, since we are at rest with respect to each one) Similarly when the clock at the origin reads t, the clock in the primed coordinate system at any point x reads

$$t' = \frac{t - vx/c^2}{\sqrt{1-v^2/c^2}} \; . \tag{3}$$

It is very convenient to define a coordinate $\tilde{t} \equiv ct$, so that time is measured in the same units as length. In terms of this coordinate and the dimensionless velocity $\beta = v/c$, these transformations can be summarized as

$$\begin{aligned} \tilde{t}' &= \frac{1}{\sqrt{1-\beta^2}}\tilde{t} - \frac{\beta}{\sqrt{1-\beta^2}}x \\ x' &= \frac{1}{\sqrt{1-\beta^2}}x - \frac{\beta}{\sqrt{1-\beta^2}}\tilde{t} \\ y' &= y \\ z' &= z \; . \end{aligned} \tag{4}$$

These transformations are known as Lorentz transformations. They are coordinate transformations on a four-dimensional spacetime, and in a moment we will reconsider them from a geometrical perspective. In the rest of this file, we will always use $\tilde{t}$ as our coordinate instead of t unless explicitly stated otherwise, so we drop the tildes.

It is often useful to draw diagrams of physical systems in spacetime. In such a diagram, a point particle appears as the line (its "world-line") which it sweeps out through spacetime. A particle at rest appears as a vertical line ($x = const.$), and a particle moving at the speed of light follows the curve $x = t$, a line at a 45° angle. Any such diagram contains an implicit choice of observer, since one draws axes x and t rather than x' and t'; the rest of this file will be concerned with how one moves between the two and what one can learn from so doing.

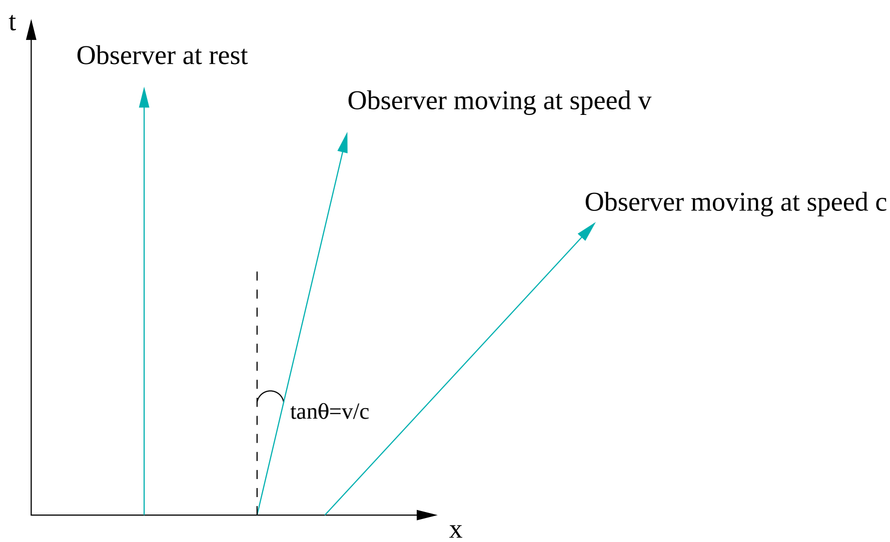

FIG. 1. A sample spacetime diagram

II. PRÉLUDE 2: ROTATIONS AND COORDINATE TRANSFORMATIONS

This section is necessarily slightly technical. The purpose is to compare Lorentz transformations to rotations; the natural language for this is matrices and vectors.

Let us compare the Lorentz transformations (4) to another familiar sort of coordinate change, a rotation in three dimensions;

$$x' = \cos\theta x + \sin\theta y$$
$$y' = \cos\theta y - \sin\theta x$$
$$z' = z \ . \tag{5}$$

In terms of the vector coordinate $\vec{x} = (x, y, z)$ (thought of as a column vector), this can be summarized as

$$\vec{x'} = \Lambda \vec{x} \tag{6}$$

where

$$\Lambda = \begin{pmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{pmatrix} \ . \tag{7}$$

We will always use the vector symbol $\vec{}$ to denote three-dimensional vectors. (As opposed to the four-vectors used above) Under a rotation (or for that matter, any other coordinate transformation like (6)) all vectors transform identically, being multiplied on the left by the matrix Λ.

Now, why is a rotation different from all other transformations? A rotation is *defined* to be a coordinate transformation which leaves unchanged all lengths of vectors and the angles between them – i.e., it leaves the dot product of any two vectors unchanged.

The dot product of two column vectors is $\vec{x} \cdot \vec{y} = x^T y$, where x^T is the matrix transpose of x. (You can quickly check that this gives the right dot product rule; the matrix notation is simply far more convenient for what we're about to do) The fact that a rotation leaves all dot products invariant means that, for any $\vec{x}$ and $\vec{y}$,

$$\begin{aligned} \vec{x'} \cdot \vec{y'} &= (x')^T y' \\ &= (\Lambda x)^T \Lambda y \\ &= x^T \Lambda^T \Lambda y = x^T y \ . \end{aligned} \tag{8}$$

Going from the first line to the second we used the fact that all vectors transform as $\vec{x} \to \vec{x'} = \Lambda \vec{x}$; going from the second to the third we used the fact that $(AB \cdots C)^T = C^T B^T \cdots A^T$, for any product of matrices. The final statement is just that $\vec{x'} \cdot \vec{y'} = \vec{x} \cdot \vec{y}$. If this statement is to be true for every $\vec{x}$ and $\vec{y}$, it must mean that

$$\Lambda^T \Lambda = \mathbf{1} \ . \tag{9}$$

This equation defines the set of matrices Λ which preserve dot products. It can be solved by means which are beyond the scope of this handout, so I'll just give you the answer. The most general matrix satisfying (9) is an arbitrary product of matrices of the form (7), optionally with the axes interchanged; these are just the familiar rotation matrices.

Now let us compare the rotations to the Lorentz transformations. I would like to claim that Lorentz transformations are simply rotations on an appropriately chosen four-dimensional space.

Definition: *Minkowski space* is four-dimensional space, coordinatized by $x = (t, x, y, z)$, with the dot product

$$x \cdot y \equiv x^T \eta y \ , \tag{10}$$

where η is the constant matrix

$$\eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \ . \tag{11}$$

This is a very unusual dot product, but it is the key to relativity. The minus sign in the top left of η is what distinguishes this from ordinary (Euclidean) four-space. It means, for example, that the length squared of a vector $\Delta x = (\Delta t, \Delta x, \Delta y, \Delta z)$[2] is

[2] This name is chosen to suggest that Δx might be the separation vector between two points of space.

$$||\Delta x|| = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \ . \tag{12}$$

Note that the sign of this is not necessarily positive. Vectors with positive norm squared are referred to as *spacelike*; vectors with negative norm squared are *timelike*; and vectors with zero norm are *null.* To see why this is, note that the unit vector in the time direction, $(1,0,0,0)$, has norm -1 and so is timelike; the unit vector in any space direction $(0,1,0,0)$ has norm 1 and is therefore spacelike; and vectors pointing exactly along the world-line of a particle moving at the speed of light, such as $(1,1,0,0)$, have zero norm and are therefore null. A vector does not have to be zero to have zero norm!

I would now like to claim that the set of rotations on this space – coordinate transformations which leave the dot product (10) invariant – are precisely Lorentz transformations. To see this, we can repeat the derivation leading up to equation (9) with the new dot product. (This is left as an exercise for the reader) The resulting equation (with Λ now a four-by-four matrix, of course) is

$$\Lambda^T \eta \Lambda = \eta \ . \tag{13}$$

Once again this equation can be solved by means which are beyond the scope of this handout. The general solution to this is a product of matrices of any of the following six forms:

$$\begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \cos\theta & & \sin\theta \\ & & 1 & \\ & -\sin\theta & & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta & \sin\theta \\ & & -\sin\theta & \cos\theta \end{pmatrix}$$
$$\begin{pmatrix} \cosh\alpha & \sinh\alpha & & \\ \sinh\alpha & \cosh\alpha & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \cosh\alpha & & \sinh\alpha & \\ & 1 & & \\ \sinh\alpha & & \cosh\alpha & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \cosh\alpha & & & \sinh\alpha \\ & 1 & & \\ & & 1 & \\ \sinh\alpha & & & \cosh\alpha \end{pmatrix} \tag{14}$$

The first three of these matrices are easy to interpret. They are ordinary, three-dimensional rotations of the coordinates x, y and z, which leave time invariant; clearly such rotations leave lengths such as (12) unchanged. The bottom three of these are a bit unusual, and require careful examination. These look more or less like rotations in the xt, yt, and zt planes, but they use hyperbolic functions instead of trigonometric ones.[3] They involve a new parameter α, which we call the *rapidity.* These transformations are called *boosts* in the x, y and z directions, respectively.

To see what this means, let us take the first one as an example. Under such a coordinate transformation,

$$\begin{aligned} t \to t' &= \cosh\alpha t + \sinh\alpha x \\ x \to x' &= \cosh\alpha x + \sinh\alpha t \\ y \to y' &= y \\ z \to z' &= z \ . \end{aligned} \tag{15}$$

We can rewrite these in a more convenient form by using some hyperbolic geometry. If we let

$$\beta = \tanh\alpha \ , \tag{16}$$

(here β is a new parameter; we will see in a moment that it is the same as our previous β) then dividing both sides of the identity $\cosh^2\alpha - \sinh^2\alpha = 1$ by $\sinh^2\alpha$ and rearranging a bit gives

$$\begin{aligned} \cosh\alpha &= \frac{1}{\sqrt{1-\beta^2}} \\ \sinh\alpha &= \beta\cosh\alpha = \frac{\beta}{\sqrt{1-\beta^2}} \ . \end{aligned} \tag{17}$$

[3]Hyperbolic functions and their properties are summarized in the appendix. One way to see why we get hyperbolic functions instead of trigonometric ones is to introduce an "imaginary time" coordinate $\lambda = it$. In terms of this coordinate, the length of a vector is just $\Delta x^2 = \Delta\lambda^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$, so this is just ordinary Euclidean space; the transformations which preserve this are just rotations. When we transform back from λ to t, however, we pick up an extra factor of i, which (since $\cosh\alpha = \cos i\alpha$ and so on) transforms all of the trig functions into hyperbolic ones.

In terms of this, (15) can be rewritten as

$$\begin{aligned} t' &= \frac{1}{\sqrt{1-\beta^2}}t + \frac{\beta}{\sqrt{1-\beta^2}}x \\ x' &= \frac{1}{\sqrt{1-\beta^2}}x + \frac{\beta}{\sqrt{1-\beta^2}}t \\ y' &= y \\ z' &= z \ . \end{aligned} \tag{18}$$

These are just the ordinary Lorentz transformations; equation (16) can be taken to be the definition of α in terms of the velocity β.

III. WHY THIS IS IMPORTANT

What we have shown is that the Lorentz transformations, together with ordinary rotations, are simply the set of transformations which leave the geometry of Minkowski space invariant. We can use this to rewrite the three postulates we began with in geometric language.

For the first postulate, two observers who differ only by their velocity are simply two observers who differ by a coordinate system but agree on all lengths and angles in 4-space. We can therefore rewrite this postulate as

1. The laws of physics depend only on lengths and angles in 4-space; they do not depend on one's choice of coordinate system.

The second postulate, that of the constancy of the speed of light, is actually encoded in our choice of the dot product on Minkowski space; we'll see this explicitly when we look at velocities in a moment. The third postulate is that people agree on simultaneous coincidences in space and time; in our language, this means that people agree on whether two things are or are not at the same point in spacetime. This is just to say that everyone agrees on whether or not two things intersect or touch in 4-space, or equivalently everyone agrees on what points are; they only differ in their meter sticks. Therefore all three postulates of relativity can be summed up by the one statement above.

Another reason this is important is calculational simplicity. Up to now, all of our calculations of how different observers see the same thing have depended on lengthy analyses of the motion of light between various observers. In four-dimensional language, this is replaced by two far simpler tools: Vectors and spacetime diagrams.

First, all vectors transform by the same rule, $x \to x' = \Lambda x$. Simply dotting vectors with matrices is typically much easier than tracing the trajectories of photons. We'll see several examples of how this is useful when discussing velocity below.

Second, one can draw diagrams in spacetime and use these to see what's happening in complicated systems. Whenever you are confused about what is happening in a relativistic system – and relativity is full of infamous pseudoparadoxes which seem contradictory until you look at them carefully! – a good rule is to start by drawing a spacetime diagram and see what is going on in four dimensions, where the accidents of a choice of coordinate system aren't relevant. (This is better than three dimensions, since a picture in three dimensions is a slice $t = constant$ of four-dimensional space; but which slice this is depends on one's choice of the t coordinate, so things can get confusing quickly)

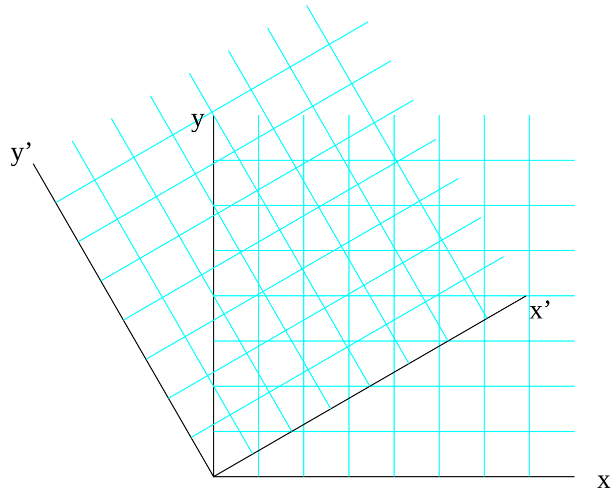

FIG. 2. Rotation of axes

Let's start by considering how Lorentz transformations look from a spacetime diagram perspective. We can compare this to rotations, shown in figure 2. For a Lorentz transformation, say Alice sees Bob moving by with velocity v along the x-axis. (Figure 3) What are Bob's axes t' and x'?

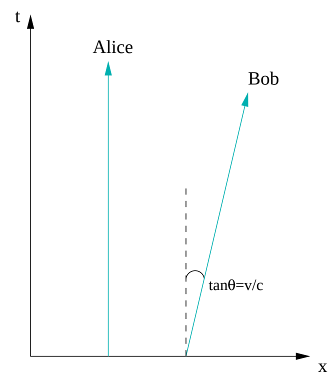

FIG. 3. Alice and Bob

Well, the time a person measures is the distance (in the sense of (12), the four-dimensional notion of distance) along his own world-line. To see this, consider Alice's own world-line, which corresponds with the time axis, or the fact that if Bob were in a sealed box (and therefore unable to see any axes set up by another observer) he would assume that he is moving vertically; the only time which he could measure, which would have any meaning whatsoever for him, is his progress along his own world-line. Therefore the t' axis coincides exactly with Bob's world-line.

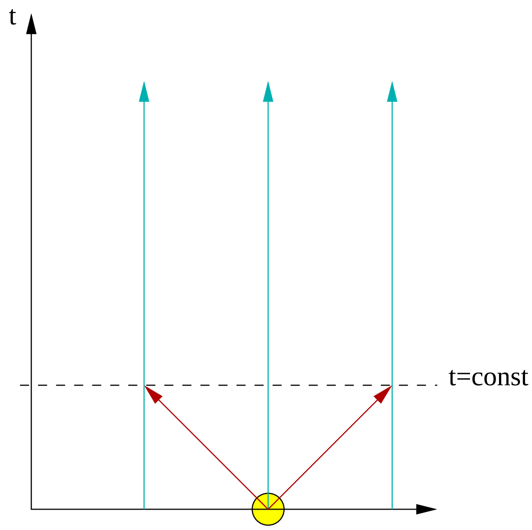

FIG. 4. Measuring the x axis, I

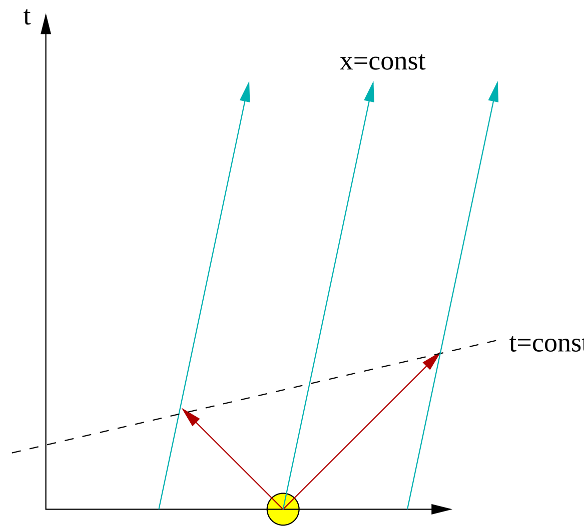

FIG. 5. Measuring the x axis, II

To see the x' axis, we can follow two different arguments, one physical and one mathematical. The physical perspective is that an observer will consider two events (this is in the technical sense, events being points in spacetime) to occur at the same time if a pulse of light emitted halfway between them hits both. In figure (4), for example, the observer concludes that the two points at which the light pulses hit the vertical lines occurred at the same value of time, and therefore lie along the axis $t = constant$, i.e. the x axis. For a moving observer, the situation is as in figure (5); his x' axis is the diagonal line formed by the two points his by his light pulses. A bit of geometry will show that this x' axis forms the same angle with the horizontal that the t' axis forms with the vertical.

The mathematical argument is to simply use the transformation formulae (15), and note that the line $t' = constant$ (i.e., the x' axis) is simply the reflection of the line $x' = constant$ (the t' axis) about the line $x = t$. Therefore Bob's space axis is the axis shown in figure 6. Note that a Lorentz transformation appears to "squash" the coordinate grid; but this is just an artifact of the way we have tried to draw Minkowski space (which is non-Euclidean!) on the Euclidean plane.

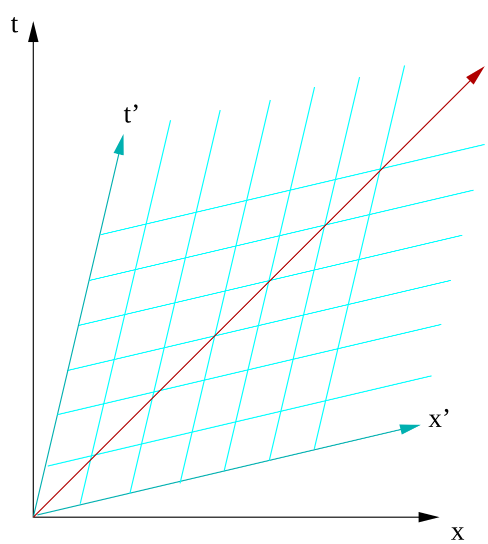

FIG. 6. Axes seen by a moving observer

Also, note that the line $x' = t'$ is the same as the line $x = t$. These lines represent an object moving at the speed of light in both frames; this means that something moving at the speed of light in one frame is moving at the speed of light in the other. Thus we see that the second postulate of relativity is in fact satisfied by our geometric construction.

In general, one can draw spacetime diagrams to represent very complicated situations. Since relativity is full of apparent paradoxes (some of which will be on future problem sets), it is very often useful to draw such a diagram to see what's going on. The advantage of such a diagram over a three-dimensional diagram is that it does not depend on one's choice of coordinate system. (A three-dimensional diagram is a diagram of a slice $t = constant$ of four-space, and which slice this is depends on one's choice of the t coordinate)

When drawing such diagrams, one can use all of the ordinary methods of determining lengths and locations, with one minor modification to take into account the fact that the time axis appears in the dot product with opposite sign. The rule is:

General Rule for Spacetime Diagrams: Whenever an angle is in a space-space plane (xy etc.), use ordinary geometry. When an angle is in a space-time plane (xt etc.), replace all trigonometric functions with their hyperbolic analogues.

This rule takes into account the fact that when you draw a right triangle in a space-time plane, the length of its hypotenuse is $-\Delta t^2 + \Delta x^2$ rather than $\Delta t^2 + \Delta x^2$. In fact, we can use this to reexamine our earlier hyperbolic identities involving the hyperbolic tangent. If we draw a right triangle with vertex angle α, (remembering now that this is a "hyperbolic" triangle, in the xt plane!) we let its base leg have length 1 and its opposite leg have length β. (See figure 7) Then $\tanh\alpha$ = opposite / adjacent = β. The length of the hypotenuse of this triangle is $\sqrt{1-\beta^2}$, and so $\cosh\alpha$ = adjacent / hypotenuse = $1/\sqrt{1-\beta^2}$ and $\sinh\alpha$ = opposite / hypotenuse = $\beta/\sqrt{1-\beta^2}$.

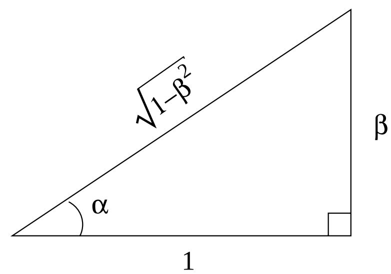

FIG. 7. Right hyperbolic triangle

To take a physics example, consider the situation of Alice holding a rod of length x oriented along the x-axis, and Bob approaching her (again along the x-axis) with speed v. Then Bob's world-line is as shown in figure 8. The length of the rod as he measures it is the extent of the rod along the x' axis; this is just the hypotenuse of a (hyperbolic) right triangle with base x and angle α, which is $x/\cosh\alpha = x\sqrt{1-\beta^2}$. This is the usual length contraction rule; all the other rules can easily be checked out using this formalism as well.

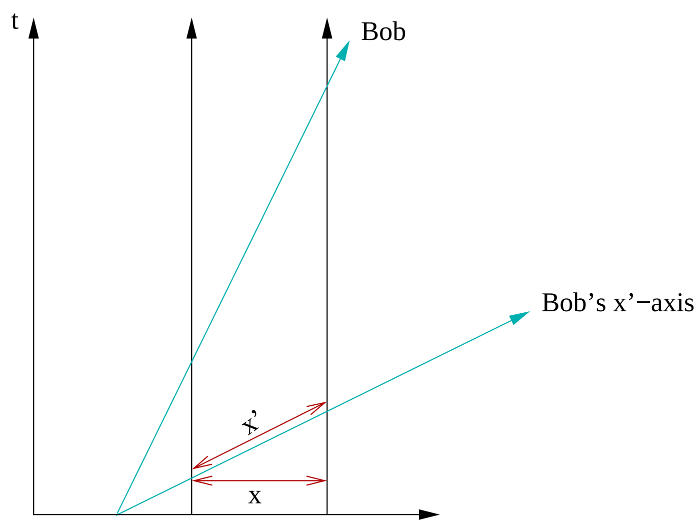

FIG. 8. Measurement of a ruler by two observers. The length of the ruler in Bob's frame is its extent along the x' axis.

Note, by the way, that if Bob were moving at the speed of light, his x' and t' axes would coincide, and so his entire coordinate system would collapse. This is not a failure of the system! What it means is that the coordinate system in Bob's rest frame is not well-defined. There is a good reason for this: by the second postulate, something moving at the speed of light must be moving at the speed of light in any frame. But Bob is always at rest with respect to his *own* rest frame; this would be a contradiction. The resolution is simply that a particle moving at the speed of light has no rest frame, and it is not meaningful to talk about the coordinate system in which it is at rest. Note that a Lorentz transformation corresponding to this situation ($\beta = 1$) would have $\alpha = \infty$, so this would correspond to an infinite boost.

The advantage of the four-dimensional formalism is therefore that it unifies all of the rules about special relativity into the simple statement that the laws of physics depend only on geometric quantities such as points, lengths and angles in four-dimensional space. Even very complicated situations can easily be studied by drawing spacetime diagrams and using the rules of hyperbolic geometry to determine the relevant quantities. Also, if physical laws are expressed in terms of four-vectors it is easy to determine how different observers measure them; all four-vectors transform the same way. We will use this principle extensively in the next section, where we consider motion and kinematics in four dimensions.

IV. THE RELATIVISTIC THEORY OF ENERGY AND MOMENTUM

In classical (Galilean) mechanics, one defines momentum to be

$$\vec{p} = m\frac{d\vec{x}}{dt} \ . \tag{19}$$

We would like to generalize this statement to four dimensions. Obviously $\vec{x}$ ought to be generalized to the four-dimensional coordinate x. The time coordinate with respect to which the derivative is taken is a non-invariant quantity, but we can naturally replace it with the *proper time* τ, defined to be the position along the object's world-line.[4] This is a useful quantity because it is invariant under Lorentz transformations; it is a four-dimensional length of the form (12). Since (in the coordinates we are using, the $\tilde{t}$'s) this length has the units of length rather than time, we should add a factor of c to make the units come out right.[5] Therefore the definition of momentum in four dimensions ought to be[6]

$$p = mc\frac{dx}{d\tau} \ . \tag{20}$$

Note that p is a vector. Under a Lorentz transformation, $x \to \Lambda x$, and (since Λ is a constant) this just means that $p \to \Lambda p$, which is just right for a vector.

To see what this means, let us evaluate it in various instances. A particle of mass m in its rest frame has world-line $x = (\tau, 0, 0, 0)$. (Its coordinate time is the same as its proper time, and it always sits at the spatial origin relative to itself) Therefore the four-momentum of an object in its own rest frame is $p = (mc, 0, 0, 0)$.

A particle moving at the speed of light has no rest frame, but we know that its world-line must be a null vector. Since the four-momentum is (up to a constant) just the tangent vector to the world-line, it should point in a null

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[4]Proper time comes from French *propre*, not any notion of propriety; it means that this time coordinate is the object's "own" notion of time.

[5]The factors of c are restored here for instructional purposes. In actual relativistic calculations that you will see elsewhere in physics, one always uses the tilded coordinates exclusively; that is, one chooses the same units for time and space so that $c = 1$. (i.e. time is measured in light-meters) This means that mass, energy and momentum all have the same units, and likewise distance and time do.

[6]One does not really have to take this on faith, but the proof goes beyond the scope of this course. In mechanics there is a general theorem (Noether's theorem) which states that to every continuous symmetry of a system there corresponds a conserved quantity; for translation invariance this quantity is momentum, and for time translation invariance this quantity is energy. If one performs the Noether's theorem calculation for Minkowski space, the four-momentum defined here is simply the invariant quantity associated with four-dimensional translation invariance. The quantity associated with four-dimensional rotation invariance is angular momentum, which is beyond the scope of this file. Note that Noether's theorem also implies that this four-momentum is a conserved quantity, a fact which we will assert without proof below.

direction; a typical four-momentum for such an object is $p = (E, E, 0, 0)$. (The four-momentum for an object moving at the speed of light still has one free parameter, which we will interpret in a moment)

Now if Alice sees a rock fly past her with velocity v, its four-momentum as she measures it will be given from its four-momentum in its own rest frame by a Lorentz transformation. (Remember that a Lorentz transformation is just a change of coordinate systems between an observer at rest, in this case the rock, and an observer in motion) She therefore sees

$$p^{(Alice)} = \Lambda p^{(Rock)} = \begin{pmatrix} \cosh\alpha & \sinh\alpha & & \\ \sinh\alpha & \cosh\alpha & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} mc\cosh\alpha \\ mc\sinh\alpha \\ 0 \\ 0 \end{pmatrix} . \tag{21}$$

Let us examine the components of this momentum in the low-velocity limit $\beta \ll 1$, so we can compare them with their nonrelativistic analogues. (This is especially important for the zeroth component, since while we have an obvious guess as to the meaning of the space components of momentum, figuring out the time component will require some ingenuity) If we Taylor expand $p^{(Alice)}$ for small β, we find

$$p_1^{(Alice)} = \frac{mc\beta}{\sqrt{1-\beta^2}} = mc\beta + \frac{1}{2}mc\beta^2 + \cdots \tag{22}$$

The first term is just mv, the ordinary momentum of the rock; the higher-order terms can be thought of as relativistic corrections to momentum. For the zeroth component of momentum,

$$\begin{aligned} p_0^{(Alice)} &= \frac{mc}{\sqrt{1-\beta^2}} = mc\left(1 + \frac{1}{2}\beta^2 + \frac{1}{8}\beta^4 + \cdots\right) \\ &= \frac{1}{c}\left(mc^2 + \frac{1}{2}mv^2 + \frac{1}{8}\frac{mv^4}{c^2} + \cdots\right) . \end{aligned} \tag{23}$$

The second term of this is recognizeably the kinetic energy, which leads us to interpret the zeroth component of momentum as E/c. The higher-order terms are relativistic corrections, like for spatial momentum. The first term is a constant independent of β, which is the energy that a particle has simply by virtue of being there. This is referred to as the "rest energy" of the particle, and what is important about it is that it is not distinct from any other kind of energy; in a collision, only the four-momentum will be conserved, so this part of it can be exchanged with kinetic energy freely.

Like any other vector, the norm of p is an invariant quantity; being the dot product of the vector with itself, all observers will agree on its value. If we evaluate it in the object's rest frame, it is easily seen to be $p^2 = -m^2c^2$; since it is invariant, it will have this value in anyone's frame. Therefore the quantity m is a relativistically invariant quantity, known as the *rest mass*, which is a characteristic of the object.

If the rest mass m is to be real, then clearly $m^2 \geq 0$. If $m \neq 0$, then this means that the norm of p is strictly negative. Since p is the tangent vector to the world-line of the object, this means that its world-line must be everywhere timelike. This is the easiest way to see that a massive object must travel no faster than the speed of light. (As a check, if a massive object were to travel at the speed of light, by equation (21) its energy would be infinite; thus an infinite amount of energy would be required to accelerate to the speed of light, confirming that it is impossible) If $m = 0$, then p is a null vector and so the particle must be travelling at the speed of light at all times; it can neither accelerate nor decelerate, only change direction. Therefore massless particles such as the photon always travel at the speed of light, and likewise only things with zero mass can do so.

Note, though, that the general expression for the momentum of a massless particle is $p = (E, E, 0, 0)$; it has one free parameter, corresponding to the energy of the particle. Special relativity makes no prediction for what this value ought to be, save that it is conserved in collisions; from quantum mechanics we know that for a photon the energy is $E = 2\pi\hbar\nu$, where $\hbar$ is Planck's constant and ν is its frequency. In general relativity this implies that photons interact gravitationally, which is the key to physical processes such as gravitational lensing.

We can now consider the example of velocity additions, which is both useful and a great example of why the four-dimensional formalism is much easier to work with than the three-dimensional. Imagine that Alice sees Bob move past her with velocity v along the x-axis, and as he does so Bob throws a rock (also along the x-axis, for simplicity) with velocity v' relative to *himself*. How fast does Alice see the rock to move? (See figure 9 for this situation from the perspective of Alice and the perspective of Bob)

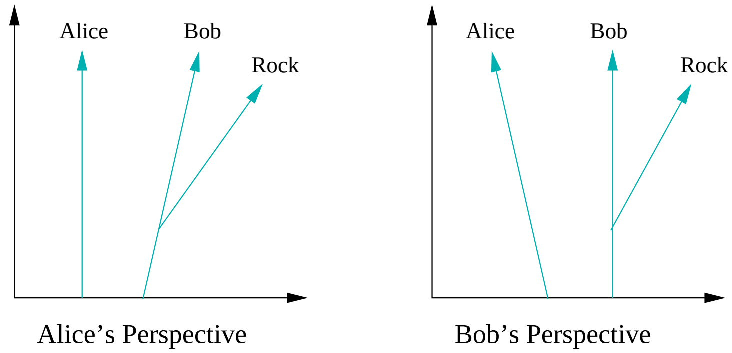

FIG. 9. Addition of velocity

We can answer this by considering how momentum transforms. The rock's momentum in its own rest frame is $p^{(rock)} = (mc, 0, 0, 0)$. Bob's frame differs from the rock's frame by a Lorentz boost along the x-axis with rapidity $\alpha' = \tanh v'/c$, and so in his frame

$$p^{(Bob)} = \Lambda_{Bob\leftarrow rock} p^{(rock)} = \begin{pmatrix} \cosh\alpha' & \sinh\alpha' & & \\ \sinh\alpha' & \cosh\alpha' & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} mc\cosh\alpha' \\ mc\sinh\alpha' \\ 0 \\ 0 \end{pmatrix} . \tag{24}$$

Alice's frame differs from Bob's frame by another Lorentz boost, this one along the x-axis with rapidity $\alpha = \tanh v/c$; so in her frame,

$$\begin{aligned} p^{(Alice)} &= \Lambda_{Alice\leftarrow Bob} p^{(Bob)} \\ &= \begin{pmatrix} \cosh\alpha & \sinh\alpha & & \\ \sinh\alpha & \cosh\alpha & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} mc\cosh\alpha' \\ mc\sinh\alpha' \\ 0 \\ 0 \end{pmatrix} \\ &= mc \begin{pmatrix} \cosh\alpha\cosh\alpha' + \sinh\alpha\sinh\alpha' \\ \cosh\alpha\sinh\alpha' + \sinh\alpha\cosh\alpha' \\ 0 \\ 0 \end{pmatrix} \\ &= mc \begin{pmatrix} \cosh(\alpha+\alpha') \\ \sinh(\alpha+\alpha') \\ 0 \\ 0 \end{pmatrix} , \end{aligned} \tag{25}$$

where the last line is a hyperbolic sum-angle identity. So Alice sees the rock moving past her with rapidity $\alpha + \alpha'$.

Note the following important properties:

1. $p^{(Alice)} = \Lambda_{Alice\leftarrow Bob} p^{(Bob)} = \Lambda_{Alice\leftarrow Bob} \Lambda_{Bob\leftarrow rock} p^{(rock)}$; that is, successive Lorentz transformations of a vector act by successive matrix multiplications.

2. Under successive boosts in the same direction, rapidities add; this is why α is a useful quantity.

We can rewrite the equation $\alpha_{Alice} = \alpha + \alpha'$ in terms of β using some hyperbolic identities and some algebra which is left as a (straightforward) exercise. The answer is

$$\beta_{Alice} = \frac{\beta + \beta'}{1 + \beta\beta'} . \tag{26}$$

This equation has several interesting properties. First, for velocities much smaller than 1, $\beta_{Alice} \approx \beta + \beta'$, which is the ordinary addition of velocities. But at large velocities, it is quite different. Say for example that $\beta' = 1$, i.e. Bob shone a flashlight in the positive x direction rather than threw a rock. Then $\beta_{Alice} = 1$ as well, so Alice sees the pulse of light receding from her at the speed of light – just the thing we wanted at the very beginning of this handout. If $\beta' = -1$, i.e. Bob shone a flashlight in the negative x direction, one again finds $\beta_{Alice} = -1$, so she agrees on the speed of light in all directions.

This equation can be generalized to the case where the motion of the rock and the motion of Bob are in different directions simply by letting the relevant Λ's be boosts in the appropriate directions. (Choose the x axis to coincide with Bob's motion relative to Alice; then a Lorentz boost in some arbitrary other direction is a rotation (to bring that other direction in line with the x-axis), a boost in the x-direction, and an inverse rotation to restore the axes. In the case where Bob threw the rock in the y direction, this reduces (check it!) to simply letting $\Lambda_{Bob\leftarrow rock}$ be a boost in the y direction, the second of the boost matrices in equation (14).) After some calculations, the general result one finds is

$$\vec{\beta}_{Alice} = \frac{\vec{\beta} + \vec{\beta}'}{1 + \vec{\beta}\cdot\vec{\beta}'} \ , \tag{27}$$

where the $\vec{\beta}$ are the *three-dimensional* velocities of the objects, and the dot product is the ordinary three-dimensional dot product.

V. RELATIVISTIC KINEMATICS

Finally, let us discuss kinematics. Although we will not prove it here (see the footnote above) the correct generalization of the conservation of energy and of momentum to special relativity is the conservation of four-momentum. As when any vector is conserved, this means that each component must be conserved independently. For example, consider the case of two physicists running at each other with relative velocity β (as seen by the first physicist), colliding and ricocheting at an angle θ in the xy-plane, also as measured by the first physicist. (See figure (10) In the rest frame of the first physicist, the initial momenta are

$$\begin{aligned} p_1 &= mc(1,0,0,0) \\ p_2 &= mc(\cosh\alpha, -\sinh\alpha, 0, 0) \end{aligned} \tag{28}$$

so the second physicist is moving with velocity $\beta = -\tanh^{-1}\alpha$. After the collision, the first physicist is moving with velocity α_1 in the stated direction, and the second is moving with velocity α_2 in the opposite direction. Thus

$$\begin{aligned} p_1' &= mc(\cosh\alpha_1, \sinh\alpha_1\cos\theta, \sinh\alpha_1\sin\theta, 0) \\ p_2' &= mc(\cosh\alpha_2, \sinh\alpha_2\cos\theta, -\sinh\alpha_2\sin\theta, 0) \ . \end{aligned} \tag{29}$$

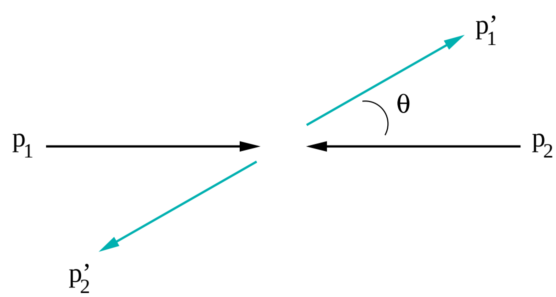

FIG. 10. Two relativistic physicists colliding. Ouch.

(We got the expressions for the space components of momentum by applying an ordinary, three-dimensional rotation to a vector pointing in the x direction; this is just the usual way a vector would be rotated.) Conservation of four-momentum therefore implies the four equations

$$\begin{aligned} mc(1+\cosh\alpha) &= mc(\cosh\alpha_1 + \cosh\alpha_2) \\ -mc\sinh\alpha &= mc\cos\theta(\sinh\alpha_1 + \sinh\alpha_2) \\ 0 &= \sin\theta(\sinh\alpha_1 - \sinh\alpha_2) \\ 0 &= 0 \ . \end{aligned} \tag{30}$$

The third equation implies that $\alpha_1 = \alpha_2$, so the physicists recoil with equal and opposite velocities. (Like in nonrelativistic collisions) The first equation then implies $\cosh\alpha_1 = (1 + \cosh\alpha)/2$, giving the final velocity; the second equation gives $\cos\theta = -\sinh\alpha/2\sinh\alpha_1$, for the recoil angle.

It is very often convenient to work in the *center-of-mass frame*, which is defined to be the frame in which the total initial momentum is aligned in the time direction. (This is the frame in which the center of mass is initially at rest; remember that the momentum of the center of mass is the sum of all the momenta)

For another example, we can prove that an electron cannot emit a single photon. (It must emit at least two photons at a time) To see this, consider the process $e \to e + \gamma$. If we work in the frame in which the electron is initially at rest, so $p_1 = (mc, 0, 0, 0)$, in the final state the electron must be moving with some velocity β in a direction which we define to be the x-direction, and the photon must be emitted with some energy and moving in the opposite direction. Thus

$$\begin{aligned} p'_e &= mc(\cosh\alpha, -\sinh\alpha, 0, 0) \\ p'_\gamma &= (E, E, 0, 0)\ . \end{aligned} \tag{31}$$

Applying conservation of momentum, we get the equations

$$\begin{aligned} mc &= mc\cosh\alpha + E \\ 0 &= E - mc\sinh\alpha\ . \end{aligned} \tag{32}$$

This implies that $\cosh\alpha + \sinh\alpha = 1$, so $\alpha = 0$, $E = 0$, and so there is no emission at all! (Or put another way, there is no solution to the momentum conservation equations with nonzero momentum transfer)

Many similar examples can be worked out, and all of relativistic kinematics amounts essentially to the application of the conservation of four-momentum to various physical situations. Since this is basically an issue of the repeated transformation and comparison of four-vectors, it falls naturally into the four-dimensional formalism developed in this file. (Take it from experience: To do kinematic calculations in the three-dimensional formalism, by deriving various rules by which momenta and energies transform, is an exercise in masochism and is remarkably physically unilluminating)

There are many other aspects to the four-dimensional formalism which we have not covered here, such as angular momentum and the construction of general Lorentz-invariant quantities beyond the rest mass. Many of these are naturally developed at a more advanced level, in the context of the Lagrangian formulation of classical mechanics; however, the concepts in this file (Minkowski space, Lorentz transformations as rotations, hyperbolic geometry, the transformation of vectors such as momenta) are the foundations of all of these notions. This is the formalism in which relativity is ordinarily studied.

VI. APPENDIX: HYPERBOLIC FUNCTIONS

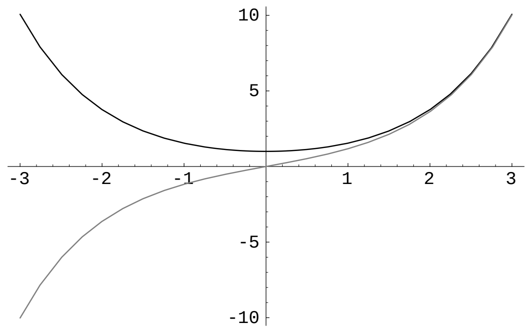

FIG. 11. The hyperbolic cosine (even) and sine (odd) functions

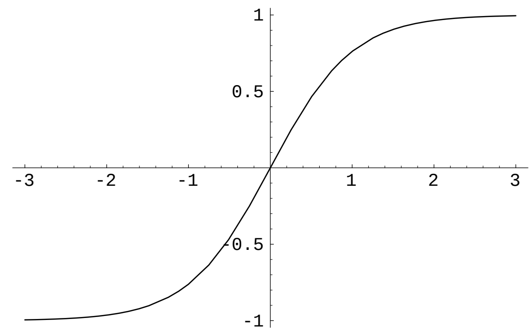

FIG. 12. The hyperbolic tangent function

The hyperbolic functions generalize trigonometric functions to Minkowski space. They are defined by

$$\cosh\alpha = \tfrac{e^{\alpha}+e^{-\alpha}}{2} \qquad\qquad \sinh\alpha = \tfrac{e^{\alpha}-e^{-\alpha}}{2}\ . \tag{33}$$

Other functions such as the hyperbolic tangent are defined in the usual ways, e.g. $\tanh\alpha = \sinh\alpha/\cosh\alpha$. These functions can be thought of as trigonometric functions of imaginary argument;

$$\cosh\alpha = \cos i\alpha \qquad\qquad \sinh\alpha = -i\sin i\alpha \tag{34}$$

and various identities corresponding to the trigonometric identities can be derived from them. Some of these identities are given below; the interested reader should probably acquire their own table of integrals and mathematical formulae, including a more detailed table of hyperbolic identities. (I use Dwight. [3])

$$\cosh^2\alpha - \sinh^2\alpha = 1 \tag{35}$$

implies a host of other identities, such as $\text{sech}^2\alpha + \tanh^2\alpha = 1$. These can all be derived by drawing a right hyperbolic triangle as in the example above. Such triangles can also be used to derive identities for mixed hyperbolic and inverse hyperbolic functions; for example, one can evaluate $\tanh\cosh^{-1}x$ by drawing a right hyperbolic triangle whose vertex angle α has hyperbolic cosine x. (Figure 13) This triangle has hypotenuse 1 and base x, so its opposite leg has length $\sqrt{1+x^2}$; thus $\tanh\alpha$=opposite/adjacent=$\sqrt{1+x^{-2}}$.

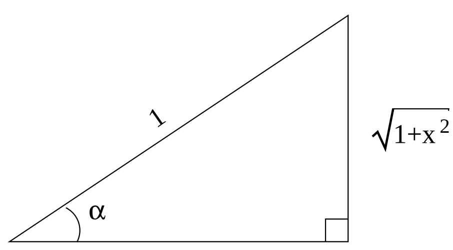

FIG. 13. Right hyperbolic triangles can be used to compute mixed hyperbolic–inverse hyperbolic identities. In this figure $\cosh\alpha = x$.

The sum angle identities for hyperbolic functions are

$$\begin{aligned}\cosh(\alpha+\alpha') &= \cosh\alpha\cosh\alpha' + \sinh\alpha\sinh\alpha' \\ \sinh(\alpha+\alpha') &= \cosh\alpha\sinh\alpha' + \sinh\alpha\cosh\alpha' \\ \tanh(\alpha+\alpha') &= \frac{\tanh\alpha + \tanh\alpha'}{1+\tanh\alpha\tanh\alpha'}\ .\end{aligned} \tag{36}$$

Inverse hyperbolic functions can be written explicitly in terms of logarithms. The simplest ones are

$$\begin{aligned}\cosh^{-1}x &= \log\left(x+\sqrt{x^2-1}\right) \\ \sinh^{-1}x &= \log\left(x+\sqrt{x^2+1}\right) \\ \tanh^{-1}x &= \frac{1}{2}\log\frac{1+x}{1-x}\ .\end{aligned} \tag{37}$$

Note that since the hyperbolic functions are not oscillatory, their inverse functions are unique. (Up to a sign in the case of cosh) The derivatives of the hyperbolic functions are

$$\begin{aligned}\partial_\alpha \sinh\alpha &= \cosh\alpha & \qquad \partial_x \sinh^{-1}x &= (x^2+1)^{-1/2} \\ \partial_\alpha \cosh\alpha &= \sinh\alpha & \qquad \partial_x \cosh^{-1}x &= (x^2-1)^{-1/2} \\ \partial_\alpha \tanh\alpha &= \mathrm{sech}^2\alpha & \qquad \partial_x \tanh^{-1}x &= (1-x^2)^{-1}\ .\end{aligned} \tag{38}$$

Further identities and properties can be found in any table of mathematical functions.

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[1] E. F. Taylor and J. A. Wheeler, *Spacetime physics: an introduction to special relativity,* 2nd ed.; W. H. Freeman, New York, 1992

[2] N. D. Mermin, *Space and time in special relativity,* McGraw-Hill, New York, 1968

[3] H. B. Dwight, *Table of integrals and other mathematical data,* 4th ed.; Macmillan, New York, 1961