Physics 124 – Problem Set # 9

(due Wednesday, December 4)

(Problem 2 is cribbed from Jackson.)

1. In class, we studied the Airy function, defined by the integral

$$\mathrm{Ai}(z) = \int_{-\infty}^{\infty} du e^{iuz + iu^3/3} \tag{1}$$

Derive further properties of the Airy function:

(a) Show that $\mathrm{Ai}(z)$ satisfies the differential equation

$$\left[\frac{d^2}{dz^2} - z\right] \mathrm{Ai}(z) = 0 \tag{2}$$

(b) Let $k(z) = (1/\sqrt{z})\mathrm{Ai}(z)$, and substitute $x = \frac{2}{3}z^{3/2}$. Show that k as a function of x satisfies the modified Bessel equation

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - (1 + \frac{\nu^2}{x^2})\right] k(x) = 0 \tag{3}$$

for the value $\nu = \frac{1}{3}$. This implies that, if $k(x) \to 0$ as $x \to \infty$, $k(x)$ must be proportional to the Hankel function $K_{1/3}(x)$.

(c) Find the constant of proportionality and the explicit relation between $K_{1/3}(x)$ and $\mathrm{Ai}(z)$ by comparing their asymptotic forms.

(d) Evaluate $\mathrm{Ai}(0)$.

(e) Let $z \to e^{i\pi}z$ carefully, evaluate $K_{1/3}(x)$ for imaginary argument, and find the expression of $\mathrm{Ai}(z)$ for negative z in terms of Bessel functions.

2. Consider the synchrotron radiation from the Crab nebula. Electrons with energies up to 10^{13} eV move in a magnetic field of the order of 10^{-4} gauss.

(a) For $E = 10^{13}$ eV, $B = 3 \times 10^{-4}$ gauss, calculate the orbit radius ρ, the cyclotron frequency Ω, and the critical frequency ω_c. What is the energy $\hbar\omega_c$ in keV?

(b) Show that for a relativistic electron of energy E in a constant magnetic field, the power spectrum of synchrotron radiation can be written

$$P(E, \omega) = \mathrm{const} \cdot \left(\frac{\omega}{E^2}\right)^{1/3} f(\frac{\omega}{\omega_c}) \tag{4}$$

where $f(x)$ is a cutoff function having the value unity at $x = 0$ and vanishing rapidly for $x \gg 1$, and $\omega_c = (eB/m)\gamma^2 \cos\theta$, where θ is the pitch of the helical path.

(c) If electrons are distributed in energy according to the spectrum $N(E)dE \sim E^{-n}dE$, show that the synchrotron radiation has the power spectrum

$$P(\omega)d\omega \sim \omega^{-\alpha}d\omega \tag{5}$$

where $\alpha = (n-1)/2$.

(d) Observations on the radiofrequency and optical continuous spectrum from the Crab nebula show that in the frequency interval from $\omega \sim 10^8$ sec^{-1} to $\omega \sim 6\times10^{15}$ sec^{-1} the constant $\alpha \approx 0.35$. At frequencies above 10^{18} sec^{-1}, the spectrum falls steeply with $\alpha > 1.5$. Determine the index n for the electron energy spectrum, as well as an upper cutoff for that spectrum. Is this cutoff consistent with the numbers of part (a)?

(e) The half-life of a particle emitting synchrotron radiation is defined as the time taken for it to lose one-half of its initial energy. What is the half-life obtained from the numbers in part (a)? How does this compare to the known lifetime of the Crab nebula? Must the energy electrons be continually replenished? From what source?