Physics 124 – Problem Set # 7

(due Wednesday, November 20)

(Problems 1–4 are cribbed from Heald and Marion.)

1. For a given distance Z, we can mark off the Fresnel zones on a screen, with the nth zone having outer radius $\rho_n = \sqrt{n\lambda Z}$. A *zone plate* is a screen made by blackening the even-numbered zones. When plane monochromatic waves are incident normally on the plate, the contributions from the transparent zones all arrive in phase at a point P on the axis at the distance Z from the plate. At observation points slightly displaced in any direction from P, the screen pattern scrambles the proper Fresnel zones, and the intensity is much less. In effect, the zone plane focuses the incident beam at P like a lens.

   (a) Compare the intensity at P produced by the zone plate with that for an aperture exposing just one Fresnel zone and with that for a very large aperture.

   (b) Show that there is actually a sequence of focal points on the axis of the zone plate. Where are they? What is their relative intensity?

   (c) Show that a zone plate focuses a point source at the distance Z_0 into a point image at the distance Z_1 in accord with the elementary lens equation

   $$\frac{1}{Z_0} + \frac{1}{Z_1} = \frac{1}{f} \tag{1}$$

   where the focus length f is the distance Z used to mark off the zones.

2. Consider a slit aperture illuminated with an amplitude that varies over the slit. Find the corresponding Fraunhofer diffraction pattern for the following cases:

   (a) $\psi_0(y) = \cos(\pi y/a)$, for $-a < y < a$.

   (b) $\psi_0(y) = \cos^2(n\pi y/a)$, for $-a < y < a$. Sketch the pattern for $n = 1$ and $n = 5$.

   (c) $\psi_0(y) = \exp(-y^2/2a^2)$, for $-\infty < y < \infty$.

3. The human eye has an aperture of diameter about 6 mm when dilated in dim light. An astronomical telescope might have a diameter of 60 cm. Choosing 5000 Å (green) as a typical wavelength of visible light, estimate the distance at which each of these can resolve a double-star system whose separation is equal to the earth-sun separation.

4. Compare optical and electron microscopes operating with photons and electrons at the same energy. The appropriate wavelength for an electron is the de Broglie wavelength, $\lambda = h/p$, where h is Planck's constant and p is the momentum of the electron. Work out these two wavelengths for a typical energy of 1 eV, and show more generally that, as long as the electrons are non-relativistic, electrons always have the advantage.

5. This problem involves a Java applet that computes the radiation patterns of an array of antennae. The setup is as follows: The antennae are assumed to be oriented in the vertical ($\hat{z}$) direction, perperdicular to the computer screen. We are interested in the antenna pattern in the ($\hat{x}, \hat{y}$) plane. So please ignore the dependence of the radiation pattern on the angle from the $\hat{z}$ axis and consider each antenna as a point radiator emitting a wave
$$\psi_0(\vec{r}) = \frac{A}{|\vec{r} - \vec{r}_i|} e^{i\alpha_i} e^{ik|\vec{r}-\vec{r}_i|} \ , \tag{2}$$
where $\vec{r}$ and $\vec{r}_i$ are two-dimensional vectors, A is an overall constant that scales out and can be omitted, and α_i is the phase of the voltage applied to the antenna.

The applet has the appearance of two boxes. The box on the left is the near field. By clicking on a button with a phase, it is possible to position antennae in this box with assigned discrete phases. The size of the box is 5 wavelengths, so that the fiducial marks are separated by $\lambda/2$. The box in the right is the far field. When the applet is working, you can click 'Compute' and this box will display the radiation intensity at large distances in some arbitrary units. The slidebar controls the step size of the grayscale used in this box. To use the slidebar, move its position and then click 'Compute' again.

(a) Show that the radiation intensity at a point $\vec{r}$ is proportional to
$$\left| \sum_i e^{i\alpha_i} e^{-ik\hat{r}\cdot\vec{r}_i} \right|^2 \tag{3}$$
Write the real and imaginary parts of the expression to be squared.

(b) Download from the class website the files:

`Antennae.html, Antennae.java, AntennaeGUI.java, PhysicsApplet.java.` (4)

Edit `Antennae.html` to remove the phrase: `archive="Antennae.jar"`. Modify the file `Antennae.java` to implement the algorithm of part (a). You should access the antenna positions and phases through the functions

`xposition(k) ,   yposition(k) ,   phase(k)` (5)

which return, respectively, the x and y coordinates of the `k`th antenna, in units of λ, and the phase of the antenna in radians. Compile the edited file, and you should have a working applet that computes antenna patterns.

(c) Compute the antenna patterns for arrays of 2, 3, 4 antennae arranged along a line with a spacing of λ between adjacent antennae. Does this agree with what we computed in class?

(d) Draw a ‘phased array’, a line of antennae with phase increasing linearly down the line. Show that this leads to a maximum of the radiation pattern at some nontrivial angle. Find a condition for the angle.

(e) Design an array of 6 antennae with as much of the energy as possible beamed along the $\hat{y}$ axis.

Hand in your code for the `Antennae` class, your solution to (d), and illustrative plots from (c), (d), and (e).

```
import java.awt.*;
import java.awt.event.*;
import java.applet.Applet;

public class Antennae extends AntennaeGUI {

  void solve(){
    resetArrays();   /* zeros the array phi */
    for (int i = 1; i <= Nx; i++){
      for (int j =1; j <= Ny ; j++){
  /* assign physical locations to point on the x,y grid */
  double x = 0.05 *(i-50);
          double y = -0.05 *(j-50);
          double R = Math.sqrt(x*x + y*y);
          if (R < 0.2) continue;  /* exit if R is too small */
          double cx = x/R;
          double cy = y/R;
          double realamp = 0.0;
          double imamp = 0.0;
          for (int k = 1; k <= nantennae; k++){
    /* fill in
           realamp +=  (something);
           imamp += (something else);
        using as ingredients:
           xposition(k) =  x position of kth antenna in units of wavelength
           yposition(k) =  y position of kth antenna in units of wavelength
           phase(k) =  phase of kth antenna in radians      */
          }
          phi[i][j] = (realamp*realamp + imamp*imamp)/(R*R);
      }
    }
    Legend.write(" ");
    refreshPicture();
  }

}
```

Figure 1: The source file `Antennae.java`.