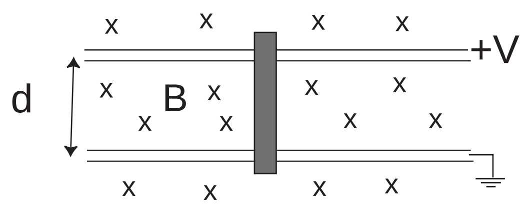
Physics 124 – Problem Set # 3

(due Friday, October 18)

1. Griffiths, problem 12.42.
2. Griffiths, problem 12.43.
3. A rail gun is formed by a conducting bar of mass m and resistance R spanning two tracks separated by a distance d and a potential difference V, with the whole system immersed in a magnetic field B. For definiteness, the top rail in the figure is at postive potential, and the magnetic field points into the paper.



   (a) Compute the force on the bar, assuming that the bar starts from rest and moves only nonrelativistically.

   (b) Now assume that the bar attains a relativistic velocity v. I do not recommend that you try to work out the force on the bar in the lab frame, since it is rather obscure how resistance transforms. Rather, work in the frame in which the bar is instantaneously at rest. Compute the $\vec{E}$ and $\vec{B}$ fields acting on the bar in this frame. Compute the force acting on the bar (in the direction parallel to the rails) in this frame. Note that it is relevant whether the largest contribution to these fields comes from the boost of the original $\vec{E}$ field or the boost of the original $\vec{B}$ field. Assuming that the original configuration had 1000 V across 1 cm in a 1000 gauss magnetic field, which source dominates in a highly boosted frame?

   (c) Return the force to the original lab frame and write the equation for dv/dt in this frame. Sketch the form of the solution for $v(t)$. You should find that v tends to a limiting asymptotic value. Compute this value for the conditions just mentioned (1000 V across 1 cm, 1000 gauss).

4. Why is it that Nature seems to contain fields $F^{\mu\nu}$ that are two-index antisymmetric tensors but not fields $H^{\mu\nu\lambda}$ that are three-index totally antisymmetric tensors? Let us explore this issue:

   (a) How many independent components does $H^{\mu\nu\lambda}$ have?

   (b) A reasonable set of Maxwell equations for the H field is

   $$\partial_\mu H^{\mu\nu\lambda} = 0 \qquad \epsilon^{\mu\nu\lambda\sigma}\partial_\mu H_{\nu\lambda\sigma} = 0 \tag{1}$$

   Look for wave solutions to these equations, of the form

   $$H^{\mu\nu\lambda} = \text{Re } \eta^{\mu\nu\lambda} e^{-ik\cdot x} \tag{2}$$

   where $k \cdot x = \omega t - \vec{k} \cdot \vec{x}$ and $\eta^{\mu\nu\lambda}$ is a set of contants. Choose $\vec{k}$ parallel to $\hat{3}$ or $\hat{z}$ for simiplicity. Write out the equations for $\eta^{\mu\nu\lambda}$ in components, and find the most general solution. How many solutions are there for a given k?

   (c) Another way to approach these equations is to write

   $$H^{\mu\nu\lambda} = \epsilon^{\mu\nu\lambda\sigma} V_\sigma \ . \tag{3}$$

   This rewriting is called a *duality transformation*, and it is completely general. Explain why. Solve the first equation for V_σ in terms of a scalar field. Plug into the second equation, and find the equation of motion of this scalar field.

   (d) Finally, show that the second equation in (b) can be automatically solved by writing H in terms of derivatives of a two-index antisymmetric potential $B_{\mu\nu}$. Show that this solution has a gauge invariance parametrized by a 4-vector C_μ. Show, by plugging this formula for H into the first equation, that $B_{\mu\nu}$ obeys the wave equation up to a gauge transformation. Define an appropriate 'me gauge' that reduces this equation to the scalar wave equation.

   (e) Construct a symmetric, gauge-invariant, conserved energy-momentum tensor for H.

5. Sound waves in the atmosphere can be described as solutions of the scalar wave equation

   $$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\chi = \sigma \tag{4}$$

   where $\chi(t, \vec{x})$ represents the density or pressure excess, $\sigma(t, x)$ is a source of the waves, and c is the speed of sound. Someone sets off a bomb at $(t, \vec{x}) = (0, \vec{0})$, a point high in the atmosphere. Represent this as a source term

   $$\sigma(t, \vec{x}) = \begin{cases} A & 0 < t < T,\ r < R \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

   where A is a constant. Assume that $T \ll R/c$, so that the explosion is essentially instantaneous.

(a) Compute the resulting radiation field $\chi(t, \vec{x})$. Use the approximation $|\vec{x}| \gg R$, $t \gg R/c$.

(b) Compute the energy flow $dP/d\Omega$ through an element of solid angle at a very large radius r as a function of time t. Recall that, for the scalar wave equation

$$\vec{j}_{\mathcal{E}} = \left\{ -\kappa \frac{\partial \chi}{\partial t} \vec{\nabla} \chi \right\} \tag{6}$$

where κ is a constant.