Physics 124 – Problem Set # 2

(due Friday, October 11)

1. In class, we argued that Lorentz transformations can be built up out of infinitesimal transformations
$$\Lambda = 1 + \epsilon t \tag{1}$$
where $t^T = \eta t \eta$ and η is the Minkowski space metric with $(1, -1, -1, -1)$ as its diagonal elements. There are six linearly independent matrices satisfying this condition: the infinitesimal generators of boosts K^1, K^2, K^3, of which
$$K^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$
generates a boost in the $\hat{1}$ direction, and the infinitesimal generators of rotations J^1, J^2, J^3, of which
$$J^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{3}$$
generates a rotation about the $\hat{1}$ axis.

   (a) Show that
$$e^{\theta J^1} = \lim_{N\to\infty} (1 + \frac{\theta}{N} J^1)^N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{4}$$
   There are two methods to do this. First, write
$$e^{\theta J^1} = 1 + \theta J^1 + \frac{1}{2!}(\theta J^1)^2 + \cdots \tag{5}$$
   compute all of the terms in the series, and sum them up. Second, note that $R(\theta) = e^{\theta J^1}$ is the solution to
$$\frac{d}{d\theta} R(\theta) = J^1 R(\theta) \tag{6}$$
   with initial condition $R = 1$ at $\theta = 0$. Show that the right-hand side of (4) has the same properties.

(b) Show that

$$e^{\eta K^1} = \lim_{N\to\infty}(1+\frac{\eta}{N}K^1)^N = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

where $\gamma = \cosh\eta$, $\beta = \tanh\eta$. Use both methods from the previous part.

(c) The operator

$$J'^i = (1+\epsilon J^j)J^i(1-\epsilon J^j) \tag{8}$$

generates rotations about an axis rotated infinitesimally from the i axis about the j direction. Work out this operator explicitly by noting that

$$J'^i = J^i + \epsilon[J^j, J^i] \tag{9}$$

Compute $[J^j, J^i]$. Is this the action you expected?

(d) Compute $[J^j, K^i]$ and interpret the result.

(e) Compute $[K^j, K^i]$ and interpret the result.

2. Griffiths, problems 12.27 and 12.37.

3. A particle of mass m, at rest, decays to two particles of masses m_a and m_b, such that $m_a + m_b < m$. Using 4-momentum conservation, find the four-vectors of the two final particles

(a) in the case $m_a = m_b = 0$.

(b) in the case $m_b = 0$ with m_a nonzero.

(c) in the general case. You will encounter the function

$$\lambda(m_1^2, m_2^2, m_3^2) = [m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2]^{1/2} \tag{10}$$

4. Griffiths, example 12.9, laboriously computes the kinematics of Compton scattering in the frame in which the initial electron is at rest. There is an easier way. Write the initial electron and photon 4-vectors as

$$p_e = (mc, 0, 0, 0) \qquad p_\gamma = \frac{1}{c}(E_0, 0, 0, E_0) \tag{11}$$

Write the final photon 4-vector as

$$p'_\gamma = \frac{1}{c}(E, E\sin\theta, 0, E\cos\theta) \tag{12}$$

Compute the final electron momentum p'_e using 4-momentum conservation. Impose the condition $(p'_e)^2 = (mc)^2$ and derive Griffiths' eq. (12.57).

5. Consider the decay of a particle of mass m, in its rest frame, to three particles of mass m_a, m_b, m_c, with $m_a + m_b + m_c < m$. Define

$$x_a = \frac{2E_a}{mc^2} \qquad x_b = \frac{2E_b}{mc^2} \qquad x_c = \frac{2E_c}{mc^2} \tag{13}$$

(a) Show that $x_a + x_b + x_c = 2$.

(b) Show that the angle between $\vec{p}_a$ and $\vec{p}_b$ can be computed in terms of the x_i. Use

$$(m_c c)^2 = p_c^2 = (P - p_a - p_b)^2 \tag{14}$$

where $P = (mc, 0, 0, 0)$.

(c) If $m_a = m_b = m_c = 0$, find the region of the x_a, x_b space allowed by 4-momentum conservation.

(d) If $m_a > 0$ but $m_b = m_c = 0$, find the region of the x_a, x_b space allowed by 4-momentum conservation.