Physics 124 – Final Exam
Solutions

1.) The Fraunhofer diffraction pattern of a circle is proportional to $|a|^2$, where

$$a = \int_{|y|<a} d^2y \; e^{-i\vec{k}\cdot\vec{y}} = \pi a^2 \left[\frac{2J_1(u)}{u}\right]$$

where $u = ka\sin\theta$

for three circles with centers at $y_x = (-3a,\ 0,\ 3a)$

$$a = \int_{|y|<a} d^2y \; e^{-i\vec{k}\cdot\vec{y}} \left[e^{3ik_x a} + 1 + e^{-3ik_x a} \right]$$

$$= \left[\pi a^2 \frac{2J_1(u)}{u}\right] \cdot e^{3ik_x a} \left(\frac{1 - e^{-i9k_x a}}{1 - e^{-i3k_x a}} \right)$$

$$|a|^2 = (\pi a^2)^2 \left| \frac{2J_1(u)}{u} \right|^2 \cdot \left| \frac{\sin\left(\frac{9}{2}k_x a\right)}{\sin \frac{3}{2}k_x a} \right|^2$$

$$u = (k_x^2 + k_y^2)^{\frac{1}{2}} a$$

The factor $\frac{2J_1(u)}{u}$ has zeros at the zeros of J_1

$$u = 3.8,\ 7.0$$

From here on, consider small angles:

$$k_x = k\frac{x}{z} \qquad k_y = k\frac{y}{z}$$

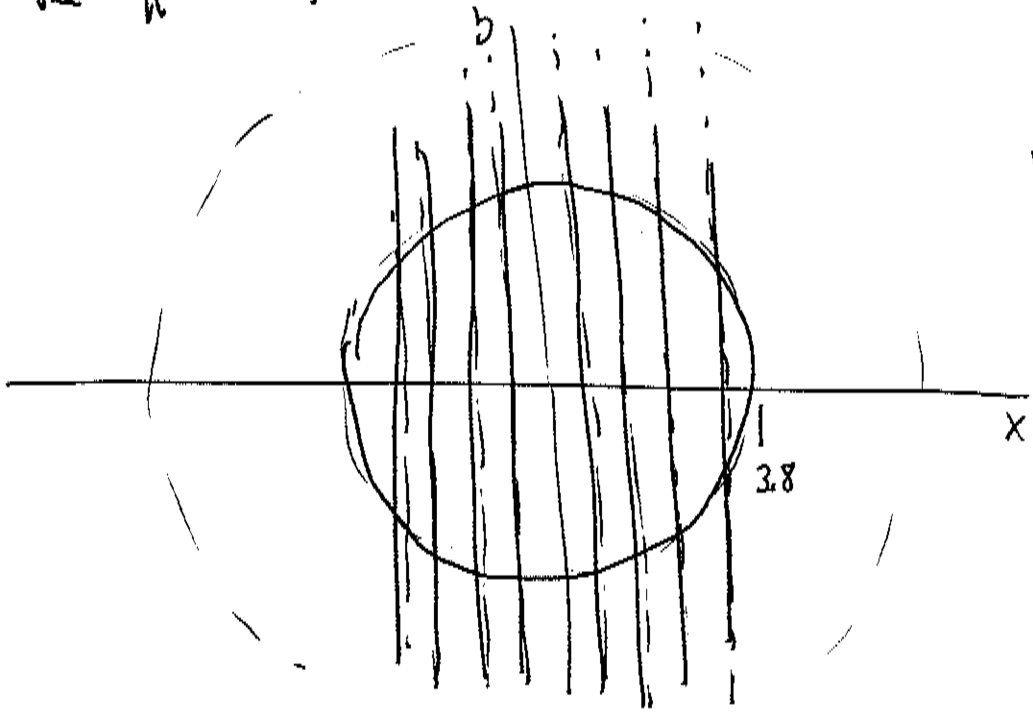
$$\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = \frac{1}{ka}\cdot(3.8,\ 7.0)$$

The factor $\left|\frac{\sin\left(\frac{9}{2}k_x a\right)}{\sin \frac{3}{2}k_x a}\right|^2$ has zeros at $\frac{9}{2}k\frac{x}{z}a = \pi n$

where n is not a multiple of 3.

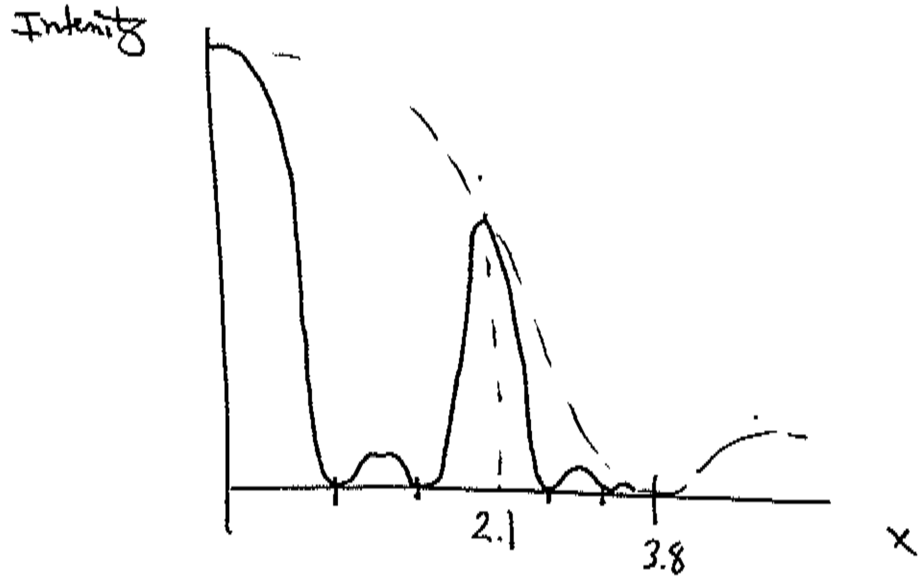
$$\frac{x}{z} = \frac{1}{ka}\cdot\Big(\underbrace{\frac{2\pi}{9}}_{0.70},\ \underbrace{\frac{4\pi}{9}}_{1.40},\ \underbrace{\frac{8\pi}{9}}_{2.8},\ \underbrace{\frac{10\pi}{9}}_{3.5}\ \cdots\Big)$$

so the diffraction pattern looks like



with zeros on the lines shown.

along the x axis:



2.) We need the wavelength for a 1 GeV proton.

$$E = 1 \text{ GeV} + mc^2 = 1938 \text{ MeV}$$

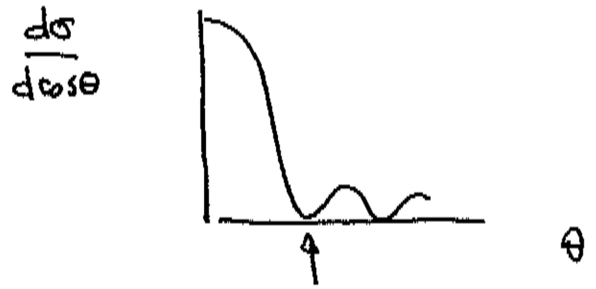
$$(pc)^2 = E^2 - (mc^2)^2 = (1938)^2 - (938)^2$$

$$= 1700. \text{ MeV}$$

$$k = \frac{pc}{\hbar c} = \frac{1700 \text{ MeV}}{197 \text{ MeV fm}} = 8.6 \text{ fm}^{-1}$$

$$\lambda = 2\pi/k = 0.73 \text{ fm}$$

a.) If the diffraction pattern is that of a sphere of radius $a = 7 \times 10^{-15}$ m $= 7$ fm, then the scatt cross sectn $d\sigma/d\cos\Theta \sim |a|^2$



with the 1st zero at $ka\sin\theta = 3.83$ = zero of J_1

$$\sin\Theta = \frac{3.83}{ka} = .064$$

$$(\theta \approx 3.6^\circ)$$

6.) now add to the diffracted wave ↙ should have been $\frac{m_\pi c}{\hbar}$!

$$\int_a^\infty d^2y \quad \alpha \; \frac{e^{-m_\pi c y/\hbar}}{y} \; e^{-iky\cos\phi}$$

$$= \int_a^\infty dy \, \cancel{y} \int_0^{2\pi} d\phi \quad \alpha \; \frac{e^{-m_\pi y/\hbar c}}{\cancel{y}} \; e^{-iky\cos\phi\sin\Theta}$$

$$= \alpha \int_0^{2\pi} d\phi \; \frac{e^{-\left[\frac{m_\pi}{\hbar c} + ik\sin\Theta\cos\phi\right]a}}{(m_\pi/\hbar c) + ik\cos\phi\sin\Theta}$$

For $a \neq 0$ this is a complicated integral. If you let $a \to 0$, the integral is now not ~~that~~ hard

$$\oint d\phi \; \frac{1}{a + i b \cos\phi} \qquad z = e^{i\phi} \qquad dz = i z \, d\phi$$

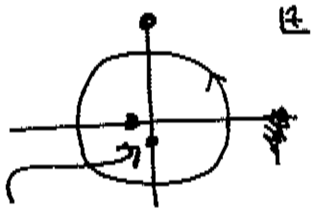
$$\cos\phi = \tfrac{1}{2}\left(z + \tfrac{1}{z}\right)$$

$$= \oint \frac{dz}{iz} \; \frac{1}{a + \frac{i}{2} b \left(z + \frac{1}{z}\right)}$$

$$= \frac{1}{-b/2} \oint dz \; \frac{1}{z^2 - i \frac{2a}{b} z + 1}$$

$$= -\frac{2}{b} \oint dz \; \frac{1}{\left[z^2 - 2 i \frac{a}{b} - \left(\frac{a}{b}\right)^2\right] + \left[1 + \left(\frac{a}{b}\right)^2\right]}$$

$$= -\frac{2}{b} \oint dz \; \frac{1}{\left(z - i\frac{a}{b} - i\left[1 + \left(\frac{a}{b}\right)^2\right]^{\frac{1}{2}}\right)\left(z - i\frac{a}{b} + i\left[1 + \left(\frac{a}{b}\right)^2\right]^{\frac{1}{2}}\right)}$$



$$z = -i\left(\left[1 + \left(\tfrac{a}{b}\right)^2\right] - \tfrac{a}{b}\right)$$

$$= -\frac{2}{b} \; 2\pi i \; \frac{1}{2 \cdot \left[-i\left(1 + \left(a/b\right)^2\right)^{\frac{1}{2}}\right]}$$

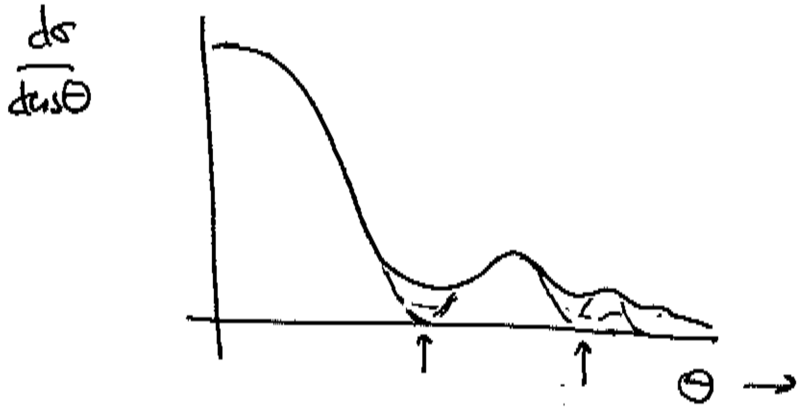
$$= \frac{2\pi}{b} \; \frac{1}{\left[1 + a^2/b^2\right]^{\frac{1}{2}}} = \frac{2\pi}{\left[b^2 + a^2\right]^{\frac{1}{2}}}$$

(but it is harder than the integral I intended to give you) !

so

$$\frac{d\sigma}{d\cos\Theta} \sim \left| \pi a^2 \frac{2J_1(ka\sin\Theta)}{ka\sin\Theta} + \frac{2\pi\alpha}{\underbrace{(m_\pi c/\hbar)^2}_{\frac{135}{197}\sim(68\,\text{fm})^2} + \underbrace{(k\sin\Theta)^2}_{(8.6\,\text{fm}^{-1}\cdot\sin\Theta)^2}} \right|^2$$

anyway, the main effect of this term is to
fill in the zeros of the diffraction pattern!



3.) The solution of

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\phi(\vec{x},t) = J(\vec{x},t)$$

is.

$$\phi(\vec{x},t) = \frac{1}{4\pi}\int d^3y \int_{t_y<t} dt_y \, \frac{1}{|\vec{x}-\vec{y}|} \, J(\vec{y},t_y)\, \delta\!\left(t-t_c-\frac{|\vec{x}-\vec{y}|}{c}\right)$$

$$\phi(\vec{x},t) = \frac{1}{4\pi}\int d^3y \, \frac{1}{|\vec{x}-\vec{y}|} \, J(\vec{y}, t - \tfrac{|\vec{x}-\vec{y}|}{c})$$

putting $\quad J = \mathrm{Re}\, J_0 \, e^{-i\omega t}$

$$= \frac{1}{4\pi} \mathrm{Re} \int d^3y \, \frac{1}{|\vec{x}-\vec{y}|} \, J_0(\vec{y}) \, e^{-i\omega(t - |\frac{\vec{x}-\vec{y}}{c}|)}$$

$$\underset{\text{in far field}}{\cong} \frac{1}{4\pi|\vec{x}|} \mathrm{Re} \int d^3y \, J_0(\vec{y}) \, e^{-i\omega t + i\frac{\omega}{c}x - i\frac{\omega}{c}\hat{x}\cdot\vec{y}}$$

$$= \mathrm{Re} \, \frac{e^{-i\omega t + ikx}}{4\pi|\vec{x}|} \int d^3y \, J_0(y) \, e^{-ik\hat{x}\cdot\vec{y}}$$

a.) the leading multipole is the monopole $\int d^3y \, J_0(\vec{y}) = \mathcal{J}_0$

there is no reason for this to vanish. So if $k|\vec{y}| \ll 1$

b.) $\phi(\vec{x},t) \cong \mathrm{Re}\left\{ \frac{e^{-i\omega t + ikx}}{4\pi|\vec{x}|} \cdot \mathcal{J}_0 \right\}$ where $\mathcal{J}_0 = \int d^3y \, J_0(\vec{y})$

c.) $\langle \vec{J}_\varepsilon \rangle \cong -\kappa \frac{(-i\omega)(+ik\,\hat{n})}{16\pi^2 r^2} \cdot \frac{1}{2} \cdot |\mathcal{J}_0|^2$

to find the power radiated, integrate this over a large sphere!

$$P = \int d^2\Omega \; r^2 \cdot \hat{r} \cdot \vec{J}_E$$

$$= 4\pi \, \kappa \frac{\omega k}{16\pi^2} \; \frac{1}{2} \; |g_0|^2$$

$$\text{Power} = \kappa \frac{\omega^2/c}{8\pi} \; |g_0|^2$$

the power emitted by the dipole, if $g_0 = 0$, is $\sim \omega^4$ as for EM radiation

[For gravitational radiation, the first nonvanishing term is the quadrupole, and $P \sim \omega^6$.]

d.) The criterion for the multipole expansion to apply is

$$k|\vec{y}| \ll 1$$

here

$$k\,|\vec{y}| \le ka = \frac{2\pi f}{c} \cdot a = \frac{2\pi \cdot \frac{\ldots}{100}/\text{sec}}{300\,\text{m/sec}} \cdot 0.25\,\text{m}$$

$$= \underline{0.524}$$

<u>marginal</u> (but radiation from higher multipoles is suppressed by numerical factors).

4.) a) $P_0 = (1-x^2)^0 = 1$

$P_1 = \frac{1}{2}(-1)\frac{d}{dx}(1-x^2) = x$

$P_2 = \left(\frac{1}{4}\right)\frac{1}{2}\frac{d^2}{dx^2}(1-2x^2+x^4)$

$= \frac{1}{8}(0-4+4\cdot 3x^2) = \frac{3x^2-1}{2}$ ✓

b.) $(-1)^\ell(1-z^2)^\ell = (z^2-1)^\ell$

The Taylor expansion of this about $z=x$ looks like

$$(z^2-1)^\ell = (x^2-1)^\ell + (z-x)\left[(x^2-1)^\ell\right]' + \cdots$$
$$+ \frac{1}{n!}(z-x)^n\left[(x^2-1)^\ell\right]^{(n)} + \cdots$$

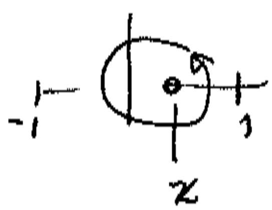
To pick out

$$\frac{(z-x)^\ell}{\ell!}\left[(x^2-1)^\ell\right]^{(\ell)}$$

we should ~~integrate~~ with $\oint \frac{dz}{2\pi i}\frac{1}{(z-x)^{\ell+1}} f(z)$

on a small contour around $z=x$

$$P_\ell(x) = \frac{1}{2^\ell}\oint \frac{dz}{2\pi i}\frac{1}{(z-x)^{\ell+1}}(z^2-1)^\ell$$



c.) For $\ell \gg 1$, write this integral as

$$\sim \frac{1}{2^\ell}\int dz\, \frac{1}{(z-x)} \exp\left[\ell\left[\log(z^2-1) - \log(z-x)\right]\right]$$

there is a saddle pt. where

$$\frac{\partial}{\partial z}(\text{exponent}) = 0$$

$$\ell\left[\frac{2z}{z^2-1} - \frac{1}{z-x}\right] = 0$$

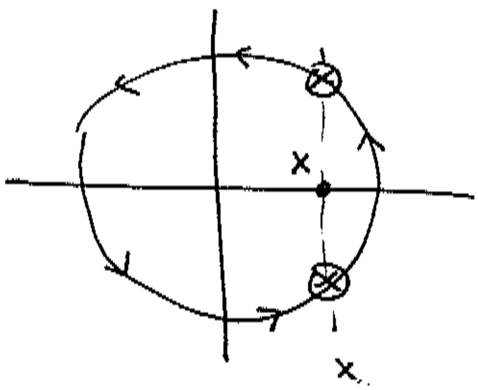
$$2z(z-x) - z^2 + 1 = 0$$

$$z^2 - 2xz + 1 = 0$$

$$(z-x)^2 = -(1-x^2) \qquad \text{w.} \quad |x|<1$$

$$z = x \pm i(1-x^2)^{1/2}$$

the saddle pts are on the unit circle



$$(z-x) = \pm i(1-x^2)^{1/2}$$

$$(z^2-1) = 2z(z-x)$$

d.) near $z = x + i(1-x^2)^{1/2}$

$$\frac{\partial^2}{\partial z^2}(\text{exponent}) = \ell\left[\frac{2}{z^2-1} - \frac{4z^2}{(z^2-1)^2} + \frac{1}{z-x}\right]$$

$$= \ell\left[\frac{2}{2z(z-x)} + 0\right]$$

$$= \frac{\ell}{z(z-x)}$$

$$= \frac{\ell}{(x+i(1-x^2)^{1/2})\, i(1-x^2)^{1/2}} = -\frac{\ell}{(1-x^2) - ix(1-x^2)^{1/2}}$$

so the integral over z is

$$\sim \int_{\text{large}}^{\text{small}} dz \quad e^{-\frac{1}{2}\ell \frac{1}{[(1-x^2)-ix(1-x^2)^{\frac{1}{2}}]}(z-x)^2}$$

$$= \underset{\text{wrong way.}}{-} \sqrt{2\pi} \frac{1}{\sqrt{\ell}} \left[(1-x^2) - i\,x(1-x^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

the contribution from $z = x - i(1-x^2)^{\frac{1}{2}}$ is

$$\int_{\text{small}}^{\text{large}} dz \quad e^{-\frac{1}{2}\ell \frac{1}{[(1-x^2)+ix(1-x^2)^{\frac{1}{2}}]}(z-x)^2}$$

$$= + \sqrt{2\pi} \frac{1}{\sqrt{\ell}} \left[(1-x^2) + i\,x(1-x^2)^{\frac{1}{2}}\right]$$

Now collect the full contribution

$$P_\ell(x) \cong \frac{1}{2^\ell}\frac{1}{2\pi i}\left\{ \left[\frac{(z^2-1)}{z-x}\right]^\ell_{\text{saddle}+} \cdot \frac{1}{(z-x)_{\text{saddle}+}} \cdot \left(-\frac{\sqrt{2\pi}}{\sqrt{\ell}}\left[(1-x^2)-ix(1-x^2)^{\frac{1}{2}}\right]\right)^{\frac{1}{2}} \right.$$

$$\left. + \left[\frac{(z^2-1)}{z-x}\right]^\ell_{\text{saddle}-} \frac{1}{(z-x)_{\text{saddle}-}} \left(+\frac{\sqrt{2\pi}}{\ell}\left[(1-x^2)+ix(1-x^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right) \right\}$$

$$\cong \frac{1}{2^\ell}\frac{1}{2\pi i}\left\{ 2^\ell\left[x+i(1-x^2)^{\frac{1}{2}}\right]^\ell \frac{1}{i(1-x^2)^{\frac{1}{2}}}\left(-\frac{\sqrt{2\pi}}{\sqrt{\ell}}\right)\left[(1-x^2)-ix(1-x^2)^{\frac{1}{2}}\right]^{\frac{1}{2}} \right.$$

$$\left. + 2^\ell\left[x-i(1-x^2)\right]^\ell \frac{1}{(-i(1-x^2)^{\frac{1}{2}})}\left(+\frac{\sqrt{2\pi}}{\ell}\right)\left[(1-x^2)+ix(1-x^2)^{\frac{1}{2}}\right]^{\frac{1}{2}} \right\}$$

$$= \frac{1}{2\pi} \frac{1}{(1-x^2)^{\frac{1}{2}}} \sqrt{\frac{2\pi}{\ell}}\, (1-x^2)^{\frac{1}{4}}$$

$$\left\{ (i)^{\ell} \left[(1-x^2)^{\frac{1}{2}} - ix \right]^{\ell+\frac{1}{2}} + (-i)^{\ell} \left[(1-x^2)^{\frac{1}{2}} + ix \right]^{\ell+\frac{1}{2}} \right\}$$

so if $e^{i\phi} = (1-x^2)^{\frac{1}{2}} + ix$

$$P_\ell(x) = \frac{1}{\sqrt{2\pi\ell}} \frac{1}{(1-x^2)^{\frac{1}{4}}} \cdot 2 \cos\left(\phi - \frac{\pi}{2}\ell \right)$$

5.) a.) $F_{\mu\nu}$ is of course gauge invariant; we need to worry about $\delta A^\mu = \partial_\mu \alpha$

$$\delta \left(\varepsilon_{\mu\nu\lambda} A^\mu F^{\nu\lambda} \right)$$

$$= \varepsilon_{\mu\nu\lambda} (\partial^\mu \alpha) F^{\nu\lambda}$$

$$\delta \int d^3x\, \varepsilon_{\mu\nu\lambda} A^\mu F^{\nu\lambda} = - \int d^3x\, \varepsilon_{\mu\nu\lambda}\, \alpha\, \partial^\mu F^{\nu\lambda}$$

$$= - \int d^3x\, \varepsilon_{\mu\nu\lambda}\, \alpha \left(\partial^\mu \partial^\nu A^\lambda - \partial^\mu \partial^\lambda A^\nu \right)$$

$= 0$ by antisymmetry.

b)

$$\delta \int d^3x \left(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} mc\, \epsilon_{\mu\nu\lambda} A^\mu F^{\nu\lambda} \right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$= \int d^3x \Big\{ -(\partial_\mu \delta A_\nu) F^{\mu\nu} + \tfrac{1}{2} mc\, \epsilon_{\mu\nu\lambda} \delta A^\mu F^{\nu\lambda} + \tfrac{1}{2} mc\, \epsilon_{\mu\nu\lambda} A^\mu (\partial^\nu \delta A^\lambda - \partial^\lambda \delta A^\nu) \Big\}$$

$$= \int d^3x \Big\{ \delta A_\nu\, \partial_\mu F^{\mu\nu} + \delta A^\mu \tfrac{mc}{2} \epsilon_{\mu\nu\lambda} F^{\nu\lambda} - \tfrac{1}{2} mc\, \epsilon_{\mu\nu\lambda} (\partial^\nu A^\mu \delta A^\lambda - \partial^\lambda A^\mu \delta A^\nu) \Big\}$$

trade names of indices

$$= \int d^3x\, \delta A^\mu \Big\{ \partial^\nu F_{\nu\mu} + \tfrac{1}{2} mc\, \epsilon_{\mu\nu\lambda} F^{\nu\lambda} + \tfrac{1}{2} mc\, \epsilon_{\mu\nu\lambda} \underbrace{(-\partial^\lambda A^\nu + \partial^\nu A^\lambda)}_{F^{\nu\lambda}} \Big\}$$

so

$$\partial^\nu F_{\nu\mu} + mc\, \epsilon_{\mu\nu\lambda} F^{\nu\lambda} = 0$$

$\epsilon^{012} = +1 \qquad \epsilon_{012} = -1 \qquad$ & cyclic c.) Now plug in

$$A^\mu = \text{Re}\ \varepsilon^\mu\ e^{-i\omega t + ikx^2}$$

we can use the gauge freedom $A^\mu \to A^\mu + \partial^\mu \alpha$ to remove ε^2 for simplicity. Then (omit (Re))

$$A^0 = \varepsilon^0\ e^{-i\omega t + ikx^2}$$

$$A^1 = \varepsilon^1\ e^{-i\omega t + ikx^2}$$

understood from here on

$$F^{01} = \partial^0 A^1 - \partial^1 A^0 = \frac{-i\omega}{c}\varepsilon^1 \cdot \left(e^{-i\omega t + ikx^2}\right)$$

$$F^{02} = \partial^0 A^2 - \partial^2 A^0 = -(-ik)\,\varepsilon^0 = ik\varepsilon^0$$

raised index on 2

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = -(-ik)\,\varepsilon^1 = ik\varepsilon^1$$

$$\partial^\nu F_{\nu\mu} + mc\,\epsilon_{\mu\nu\lambda} F^{\nu\lambda} = 0$$

$\mu=0$ $\quad \partial$ [redacted] F [redacted] $+ \partial^2 F_{20} + 2mc\,\epsilon_{012} F^{12} = 0$

$$(-ik)(+ik\varepsilon^0) - 2mc\,(ik\varepsilon^1) = 0$$

$$+ik\,\varepsilon^0 + 2mc\,\varepsilon^1 = 0$$

$\mu=1$ $\quad \partial^0 F_{01} + \partial^2 F_{21} + 2mc\,\epsilon_{120} F^{20} = 0$

$$-i\frac{\omega}{c}\left(+i\frac{\omega}{c}\varepsilon^1\right) + (-ik)(-ik\varepsilon^1) + 2mc(-1)(-ik\varepsilon^0) = 0$$

$$\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]\varepsilon^1 + ik\,2mc\,\varepsilon^0 = 0$$

$\mu = 2$ $$\partial^0 F_{02} + \partial^1 F_{12} + 2mc\ \epsilon_{201} F^{01} = 0$$

$$\left(-i\frac{\omega}{c}\right)\left(-ik\varepsilon^0\right) + 0 + 2mc\,(-1)\,\frac{-i\omega\varepsilon^1}{c} = 0$$

$$-ik\,\varepsilon^0 - 2mc\,\varepsilon^1 = 0$$ Same as $\mu = 0$ ✓

so $$ik\,\varepsilon^0 = -2mc\,\varepsilon^1$$

then

$$\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]\varepsilon^1 - (2mc)^2\,\varepsilon^1 = 0$$

so $$\left(\omega/c\right)^2 = k^2 + (2mc)^2$$

the photon gets a mass !

this Lagrangian is called "Chern - Simons theory".