Physics 124 – Final Exam

(Thursday, December 12)

There are five problems, each worth 20 points.

1. A black screen has three circular holes punched in it. The holes have radius a and they are in a line with their centers separated by $3a$. Compute the Fraunhofer diffraction pattern behind this screen. Sketch the intensity distribution. Where are its zeros?

2. Consider bombarding an atomic nucleus with fast protons. Quantum mechanics applies, so the protons should be considered as waves with wavelength $\lambda = 2\pi/k$ where the momentum $p = \hbar k$.

   (a) To first approximation, protons that hit the face of the nucleus are absorbed, while protons that miss go straight ahead. The forward scattering then has the form of a diffraction pattern. Sketch the shape of the scattering cross section $d\sigma/d\cos\theta$. For a Pb nucleus, the radius of the sphere is 7×10^{-15} m. If the Pb is bombarded with protons of energy 1 GeV, at what angle do we find the first zero of the differential cross section?

   (b) Of course, the nucleus does not really have a sharp edge. To the model of total absorption, we might add a component of partial absorption whose shape is

   $$\delta A(r) = \alpha \frac{e^{-m_\pi c r/\hbar}}{r} , \tag{1}$$

   where r is the impact parameter. Integrating this from the edge of the nucleus to infinity,[1] what contribution is added to the amplitude of the scattered wave? What is the qualitative effect on the differential cross section?

3. Consider a drum of radius a, oscillating at a frequency ω. The drum emits *sound*, which is described by a scalar wave equation. Work out the multipole radiation formula for sound, using the equation

   $$[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2]\phi(\vec{x},t) = J(\vec{x},t) \tag{2}$$

   where $J(\vec{x},t)$ is a localized scalar source. Let J be periodic:

   $$J(\vec{x},t) = \text{Re } J_0(\vec{x})e^{-i\omega t} \tag{3}$$

   (a) What is the leading multipole in the radiation formula?

   (b) What is the wave amplitude in this multipole, in terms of $J_0(\vec{x})$?

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[1] It has been pointed out that this integral is not so simple. Do the best you can.

(c) Using the expression

$$\vec{J}_{\mathcal{E}} = -\kappa \frac{\partial \phi}{\partial t} \vec{\nabla} \phi \ , \tag{4}$$

derive the formula for the power radiated.

(d) For $a = 0.25$ m, $f = \omega/2\pi = 100/\text{sec}$ (a bass note), $c = 300$ m/sec, are we in a situation where the multipole expansion applies?

4. The Legendre polynomial is given by the following formula (Rodrigues' formula):

$$P_\ell(x) = \frac{1}{2^\ell} \frac{(-1)^\ell}{\ell!} \frac{d^\ell}{dx^\ell} (1 - x^2)^\ell \tag{5}$$

We can use this formula to find the form of Legendre polynomials for very large ℓ.

(a) Work out $P_0(x)$, $P_1(x)$, and $P_2(x)$ to see that this looks familiar.

(b) Show that the above formula implies the relation

$$P_\ell(x) = \frac{1}{2^\ell} \oint \frac{dz}{2\pi i} \frac{(z^2 - 1)^\ell}{(z - x)^{\ell+1}} \tag{6}$$

where the contour encloses $z = x$.

(c) For $\ell \gg 1$, the integral can be analyzed by steepest descents. For x real and $|x| < 1$, show that the saddle points are at

$$z_\pm = x \pm i\sqrt{1 - x^2} \tag{7}$$

(d) Find the limiting form of $P_\ell(x)$ for large ℓ.

5. Consider the following Lagrangian for an electromagnetic field in 3 space-time dimensions:

$$\int d^3x \left\{ -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2} mc \epsilon_{\mu\nu\lambda} A^\mu F^{\nu\lambda} \right\} \tag{8}$$

where m is a constant and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) Show that the Lagrangian is gauge-invariant.

(b) Compute the equation of motion for A_μ.

(c) Look for wave solutions travelling in the $\hat{2}$ direction

$$A^\mu(\vec{x}, t) = \text{Re}\ \epsilon^\mu e^{-i\omega t + ikx^2} \ . \tag{9}$$

We can choose the (Coulomb) gauge where $\epsilon^2 = 0$. In this gauge, we obtain two equations for ϵ^0 and ϵ^1. Show that these equations are consistent only if $(\omega/c)^2 = k^2 + (2mc)^2$, that is, if the photon has a mass.

Some useful quantities:

$$\begin{aligned}
\epsilon_0 &= 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2} \\
\mu_0 &= 4\pi \times 10^{-7}\ \mathrm{N/A^2} \\
e &= 1.60 \times 10^{-19}\ \mathrm{C} \\
c &= 3.00 \times 10^{8}\ \mathrm{m/sec} \\
\hbar &= 1.055 \times 10^{-34}\ \mathrm{J\ sec} \\
\hbar c &= 197.3\ \mathrm{MeV\ fm} \\
m_p c^2 &= 938.2\ \mathrm{MeV\ fm} \\
m_e c^2 &= 0.511\ \mathrm{MeV\ fm} \\
m_\pi c^2 &= 135.\ \mathrm{MeV} \\
1\ \mathrm{fm} &= 10^{15}\ \mathrm{m} \\
1\ \mathrm{eV} &= 1.6 \times 10^{-19}\ \mathrm{J}
\end{aligned} \tag{10}$$

Zeros of Bessel functions $J_m(z)$:

J_0	2.405	5.520	8.654
J_1	3.832	7.016	10.173

Values of $J'_m(z)$ at these zeros:

J_0	-0.519	0.340	-0.271
J_1	-0.403	0.300	-0.250