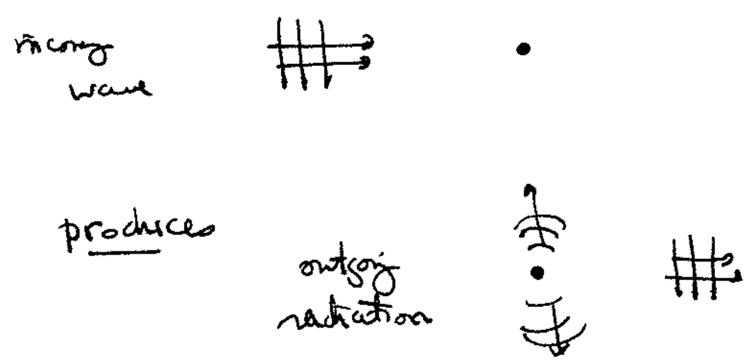
Scattering of Electromagnetic Waves

We now know how charged particles respond to electromagnetic fields and how charged particles produce electromagnetic fields. By just putting these pieces together, we can analyze the following ~~interesting~~ problem: Imagine that an electromagnetic wave is coming in along the $\hat{z}$ axis, and ~~that~~ there is a charged particle of mass m and charge q at $\vec{x}=0$. As the wave comes through, the charge will move in response to the electromagnetic fields. As a result of its acceleration, it will radiate. Then we will find



We say that the charged particle scatters the electromagnetic radiation. Let's develop the formulae that describe this physics.

Write the wave as

$$\vec{E} = \mathrm{Re}\ E_0 \vec{\epsilon}\ e^{-i\omega t + ikz} \qquad \vec{B} = \frac{\hat{k}}{c} \times \vec{E} \qquad \vec{\epsilon} \perp k\hat{z}$$

The equation of motion for the charged particle is

$$m\ddot{\vec{x}} = q\vec{E}$$

so
$$\ddot{\vec{x}} = \mathrm{Re}\ \frac{q}{m}\vec{\epsilon}E_0 e^{-i\omega t} \quad \text{at} \quad \underline{z=0}$$

$$\vec{x} = \mathrm{Re}\ \frac{q}{-\omega^2 m}\vec{\epsilon}E_0 e^{-i\omega t}$$

The particle oscillates in the field. This produces an oscillating electric dipole moment

$$\vec{p}(t) = q\vec{x}(t) \quad \Rightarrow \quad \vec{p}(t) = \mathrm{Re}\ \frac{q^2}{-\omega^2 m}\vec{\epsilon}E_0 e^{-i\omega t}$$

The radiation from this oscillating dipole moment is a $\sin^2\theta$ distribution about the axis $\vec{\epsilon}$

$$\frac{dP}{d\Omega} = \frac{\mu_0}{32\pi^2}\frac{\omega^4}{c}p_0^2\sin^2\theta$$

$$= \frac{\mu_0}{32\pi^2}\frac{\omega^4}{c}\left(\frac{q^2}{-\omega^2 m}\vec{\epsilon}E_0\right)^2\sin^2\theta$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^4}{32\pi^2 m^2 c}E_0^2\sin^2\theta$$

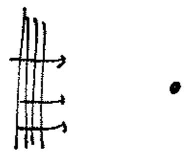
the total power is

$$P = \frac{\mu_0 q^4}{12\pi m^2 c}E_0^2$$

We can compare this to the total power incident on a surface in front of the scattering particle. This is

$$\hat{z}\cdot\langle\vec{S}\rangle = \frac{1}{2}\frac{1}{\mu_0} E_0 \cdot \frac{E_0}{c} = \frac{1}{2}\frac{E_0^2}{\mu_0 c}$$

per unit area on a plane in front of the particle.



The ratio of the scattered power to the incident power/Area is

$$\sigma = \frac{\text{scattered power}}{\text{incident power/area}} = \frac{\dfrac{\mu_0 q^4}{12\pi m^2 c} E_0^2}{\dfrac{E_0^2}{2\mu_0 c}} = \frac{\mu_0^2 q^4}{6\pi m^2}$$

Let's write

$$\mu_0 = \frac{1}{\epsilon_0 c^2}$$, then this becomes.

$$= \frac{8\pi}{3}\left(\frac{q^2}{4\pi\epsilon_0}\frac{1}{mc^2}\right)^2$$

For an elementary charged particle – an electron or a proton, we have

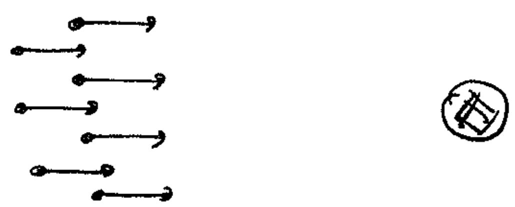
$$\sigma = \frac{8\pi}{3}\left(\frac{e^2/4\pi\epsilon_0}{mc^2}\right)^2$$

σ has the units of area:

$$\frac{e^2}{4\pi\epsilon_0} \sim \text{J}\cdot\text{m} \qquad mc^2 \sim \text{J} \qquad \text{so} \qquad \sigma \sim m^2$$

This is, in fact, the effective area of the incoming wave that the charged particle takes out and disperses as scattered radiation.

In mechanics, we meet a similar concept. Consider a situation in which a flux of particles comes in along $\hat{z}$ and intersects a large particle which scatters them:



We can define

$$\sigma = \frac{\text{rate of scattering (particles/sec)}}{\text{\# of incoming particles / m}^2\text{ sec}} = \frac{\text{rate of scattering}}{\text{flux of incoming particles}}$$

Again σ is an area (m^2), the area of the scattering center or of its cross-section. In any context σ is called the cross section. When we speak of the angular dependence of scattering or energy loss, we measure the power of the scatterer in terms of a differential cross section

$\frac{d\sigma}{d\Omega}$ or $\frac{d\sigma}{dE}$. For example:

$$\frac{\text{rate of scattering into the solid angle } d\Omega}{\text{flux of incoming particles}} = \frac{d\sigma}{d\Omega}\, d\Omega$$

In quantum mechanics, we will reinterpret an electromagnetic wave as a collection of photons, particles of electromagnetic energy with

$$E = \hbar\omega \qquad \vec{p} = \hbar K$$

If we then write

$$\text{Power} = \text{energy/sec} = \hbar\omega \cdot (\#\text{ of photons/sec})$$

Then the formula above for the cross-section of a scatterer of electromagnetic waves becomes the standard definition for the cross section of a scatterer of particles. On the other hand, we do not need to appeal to the quantum level; the purely classical definition is sufficient.

The formulas above for the cross section for the scattering of electromagnetic waves from an elementary charge ("Thomson scattering")

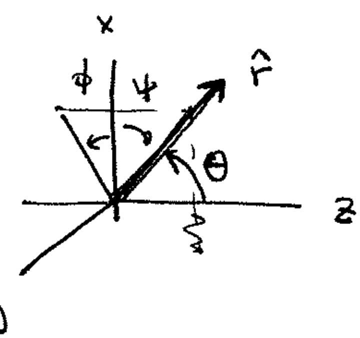
$$\sigma = \frac{8\pi}{3}\left(\frac{e^2/4\pi\epsilon_0}{mc^2}\right)^2$$

is called the Thomson cross-section. The associated unit of distance is

$$r_e = \frac{e^2}{4\pi\epsilon_0} \frac{1}{mc^2}$$

For an electron, $r_e = 2.8 \times 10^{-13}$ cm ; this is called the "classical electron radius". Actually, we know from high-energy scattering experiments that the electron is pointlike down to distances of at least 10^{-18} cm. On the other hand, the classical proton radius $r_p = \frac{e^2}{4\pi\epsilon_0} \frac{1}{m_p c^2} = 1.5 \times 10^{-16}$ cm. But the proton is a bound state of three quarks, and its actual size is about 10^{-13} cm.

Let's now compute the angular distribution for Thomson scattering. Consider first an electromagnetic wave polarized along the $\hat{x}$ axis. Define angles from the scattering center



If $\hat{r}$ is the direction of outgoing radiation, θ is the angle between $\hat{r}$ and $\hat{z}$, ψ is the angle between $\hat{r}$ and $\hat{x}$

$$\cos\theta = \hat{r}\cdot\hat{z} \qquad \cos\psi = \hat{r}\cdot\hat{x}$$

We usually write a unit vector in polar coordinates as

$$\hat{r} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

so

$$\cos\psi = \sin\theta\cos\phi$$

$$\sin^2\psi = 1 - \sin^2\theta\cos^2\phi$$

Now, when I computed the radiated power on p. 2, the distribution is actually with respect to the $\hat{x}$ axis, so

$$\frac{dP}{d\Omega} = \frac{\mu_0 e^4}{32\pi^2 m^2 c} E_0^2 \sin^2\psi$$

Dividing by the incoming flux of energy $\frac{1}{2\mu_0} E_0^2 \frac{1}{c}$,

$$\frac{d\sigma}{d\Omega} = \frac{\mu_0^2 e^4}{16\pi^2 m^2} \sin^2\psi$$

or $$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2 (1 - \sin^2\theta\cos^2\phi) \qquad \vec{\epsilon} = \hat{x}$$

similarly, for $\vec{\epsilon} = \hat{y}$

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2 (1 - \sin^2\theta\sin^2\phi) \qquad \vec{\epsilon} = \hat{y}$$

If we have unpolarized radiation, a mixture of random oriented waves, we should average over ϕ or average $\cos^2\phi$ and $\sin^2\phi$:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpolarized}} = \left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2 \left(1 - \tfrac{1}{2}\sin^2\theta\right)$$

$$= \frac{1}{2}\left(\frac{e^2/4\pi\epsilon_0}{mc^2}\right)^2 (1+\cos^2\theta)$$

Integrating

$$\sigma = \int d\Omega \, \frac{d\sigma}{d\Omega} = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \, \frac{d\sigma}{d\Omega}$$

we find again the Thomson result for the total cross section.

I should remark that Thomson's result holds only when

$$\hbar\omega \ll mc^2$$

that is, when we can ignore the recoil of the electron when it is hit by a quantum of electromagnetic radiation. When $\hbar\omega$ is comparable to mc^2, we can find from relativistic kinematics ~~that~~ the scattered photon has a lower energy, and a lower frequency, according to

$$\hbar\omega' = \frac{\hbar\omega}{1 + \frac{\hbar\omega}{mc^2}(1-\cos\theta)}$$

The process is now called Compton scattering.

The Thomson cross section is replaced by the Klein-Nishina formula:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{e^2/4\pi\epsilon_0}{mc^2}\right)^2 \left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\Theta\right]$$

This distribution is peak toward forward angles $(\Theta \to 0)$ where ω' is largest.

All of the analysis above has been done for the case in which the scatterer is a free particle. But it is easy to repeat it for the model of atomic electrons that we used in 121, in which electrons are viewed as classical particles bound to atoms by harmonic oscillators of definite frequency. We noted that the equation of motion

$$m(\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2 \vec{x}) = -e\,\vec{E}(t,\vec{x})$$

models an atomic transition with $\Delta E = \hbar\omega_0$ to an excited atomic state of lifetime $\tau = 1/\gamma$. For a wave of definite frequency, this equation becomes

$$m(-\omega^2 - i\gamma\omega + \omega_0^2)\,\vec{x}_0 = -e\,\vec{E}_0$$

or (replaying the analysis on p. 2)

$$\vec{p}(t) = \mathrm{Re}\,\frac{(e^2/m)}{(-\omega^2 - i\gamma\omega + \omega_0^2)}\,\vec{\epsilon}\,E_0\,e^{-i\omega t}$$

Again there is an oscillating electric dipole moment, and again the radiation appears in a $\sin^2\psi$ distribution about the axis of the dipole. So

$$P = \frac{\mu_0 e^4}{12\pi m^2 c} \frac{\omega^4}{|\omega^2 + i\gamma\omega - \omega_0^2|^2} E_0^2$$

$$= \frac{\mu_0 e^4}{12\pi m^2 c} \frac{\omega^4}{(\omega^2-\omega_0^2)^2 + (\gamma\omega)^2} E_0^2$$

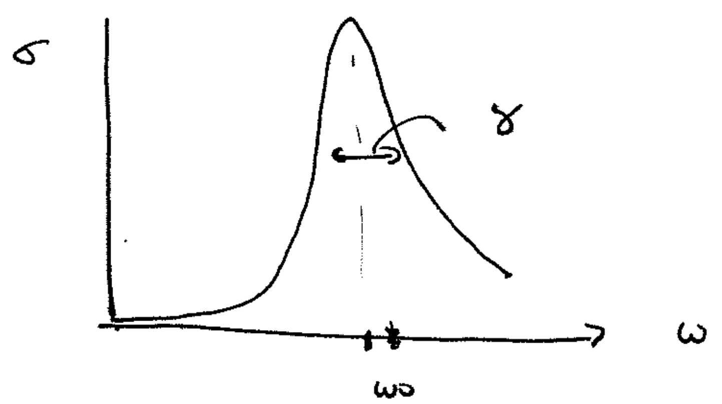
dividing out the incident flux

$$\sigma = \frac{8\pi}{3}\left(\frac{e^2/4\pi\epsilon_0}{mc^2}\right)^2 \frac{\omega^4}{(\omega^2-\omega_0^2)^2+(\gamma\omega)^2}$$

with, as before

$$\frac{d\sigma}{d\Omega} \propto (1+\cos^2\theta)$$

The form of σ is a resonance at the frequency ω_0



with the tail enhanced by the ω^4 factor.

An important special case is $\omega \ll \omega_0$. Typical atomic transition has $\Delta E \sim 10\,eV$, while visible light has $\hbar\omega = 2\text{-}3\,eV$, so when we look at the scattering of visible light by molecules, we are in this regime. For this case, called Rayleigh scattering

$$\sigma = \frac{8\pi}{3}\left(\frac{e^2/4\pi\epsilon_0}{mc^2}\right)^2\left(\frac{\omega^4}{\omega_0^4}\right)$$

Note the much stronger scattering at high frequencies or short wavelengths ($\lambda = 2\pi c/\omega$), due to the factor ω^4. Among the consequences of this factor is the fact that, when sunlight scatters from molecules in the atmosphere, the cross section is much higher for blue light than for red light. So, when we look away from the sun and view the scattered light, we see a blue sky.