Antennas

April 27

A property of the multipole expansion that we discussed in the previous lecture is the characteristic radiation pattern of each multipole:

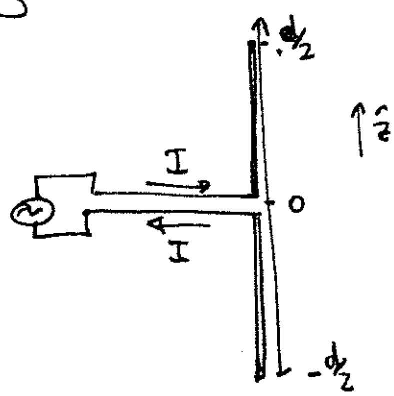
$$\frac{dP}{d\Omega} \sim \sin^2\theta \quad \text{for the electric or magnetic dipole}$$

$$\frac{dP}{d\Omega} \sim \sin^2\theta\cos^2\theta \quad \text{for a cylindrically symmetric quadrupole}$$

etc.

In many practical applications such as radio transmission, we would like to shape this pattern to need our needs. For example, we would sometimes like to beam radiation in one direction and not in another. To do this, we need an extended radiator — an antenna.

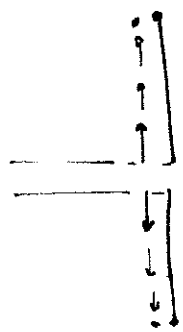
Let's analyze the simplest model of an antenna. Start with a thin wire of length d, cut it in the middle, and drive it with an alternating current at frequency ω:



A current will flow in the wire with frequency ω. Along $\hat{z}$, this current will vanish at $z = \pm d/2$ and varies along the wire.

A standard model is

$$I(z) = I_0 \sin k\left(\frac{d}{2} - |z|\right) \cos\omega t, \text{ with } k = \omega/c$$



This corresponds to a current

$$\vec{J} = \mathrm{Re}\, e^{-i\omega t} I_0 \sin k\left(\frac{d}{2} - |z|\right) \cdot \delta(x)\,\delta(y)$$

Since $\vec{J}$ is not uniform, there is also a time-dependent charge density on the wire. The $\vec{A}$ field, however, only involves $\vec{J}$ and is given by

$$\vec{A}(t,\vec{x}) = \frac{\mu_0}{4\pi} \mathrm{Re}\, e^{-i\omega t} \int d^3y \frac{\vec{J}(\vec{y})}{|\vec{x}-\vec{y}|} e^{ik|\vec{x}-\vec{y}|}$$

$$= \frac{\mu_0}{4\pi} \mathrm{Re}\, e^{-i\omega t} \int_{-d/2}^{d/2} dz \frac{I_0 \sin k\left(\frac{d}{2} - |z|\right)}{|\vec{x} - z\hat{z}|} e^{ik|\vec{x} - z\hat{z}|}$$

I would like to imagine an extended source, for which d might be of order $\lambda = 2\pi/k$, or larger. But as long as we assume that

$$|\vec{x}| \gg \lambda, d \qquad \text{with} \quad \lambda \sim d$$

we can make the "far field" approximations

$$\frac{1}{|\vec{x} - \hat{z}z|} \cong \frac{1}{x}$$

$$e^{ik|\vec{x}-\hat{z}z|} \cong e^{ikx - ikz\cos\theta}$$

$$\vec{\nabla}\, e^{ik|\vec{x}-\hat{z}z|} \cong (ik\hat{x})\, e^{ikx - ikz\cos\theta}$$

then

$$\vec{A} = \frac{\mu_0}{4\pi} \,\mathrm{Re}\, \frac{e^{-i\omega t} e^{ikx}}{x} \hat{z} \int_{-d/2}^{d/2} dz \; I_0 \sin k\left(\tfrac{d}{2} - |z|\right) e^{-ikz\cos\theta}$$

$$\vec{B} = \frac{\mu_0}{4\pi} \,\mathrm{Re}\, \frac{e^{-i\omega t} e^{ikx}}{x} (ik\,\hat{x}\times\hat{z}) \int_{-d/2}^{d/2} dz \; I_0 \sin k\left(\tfrac{d}{2} - |z|\right) e^{-ikz\cos\theta}$$

the ~~integral~~ is

$$\mathcal{I} = \int_{-d/2}^{d/2} dz \; I_0 \sin k\left(\tfrac{d}{2} - |z|\right) e^{-ikz\cos\theta}$$

$$= \int_{-d/2}^{d/2} dz \; I_0 \frac{1}{2i}\left(e^{ik\frac{d}{2}} e^{-ik|z|} - e^{-ik\frac{d}{2}} e^{ik|z|}\right) e^{-ikz\cos\theta}$$

$$= \int_{0}^{d/2} dz \; I_0 \frac{1}{2i}\left(e^{ik\frac{d}{2}} e^{-ikz} - e^{-ik\frac{d}{2}} e^{ikz}\right)\left(e^{-ikz\cos\theta} + e^{ikz\cos\theta}\right)$$

$$= \frac{I_0}{2i}\left\{\frac{(e^{-ik\frac{d}{2}\cos\theta} - e^{ik\frac{d}{2}})}{-ik(1+\cos\theta)} + \frac{e^{ik\frac{d}{2}\cos\theta} - e^{ik\frac{d}{2}}}{-ik(1-\cos\theta)}\right.$$

$$\left. - \frac{(e^{-ik\frac{d}{2}\cos\theta} - e^{-ik\frac{d}{2}})}{ik(1-\cos\theta)} - \frac{e^{ik\frac{d}{2}\cos\theta} - e^{-ik\frac{d}{2}}}{ik(1+\cos\theta)}\right\}$$

$$= \frac{I_0}{2k}\left(\frac{1}{1+\cos\theta} + \frac{1}{1-\cos\theta}\right)\left[e^{-ik\frac{d}{2}\cos\theta} + e^{ik\frac{d}{2}\cos\theta} - e^{i\frac{kd}{2}} - e^{-i\frac{kd}{2}}\right]$$

$$= -\frac{2I_0}{k}\left[\frac{\cos\frac{kd}{2} - \cos(\frac{kd}{2}\cos\theta)}{\sin^2\theta}\right]$$

then

$$\vec{B} = \frac{\mu_0}{4\pi}\,\mathrm{Re}\,\frac{e^{-i\omega t + ikx}}{x}(-2iI_0)(\hat{x}\times\hat{z})\left(\frac{\cos\frac{kd}{2} - \cos(\frac{kd}{2}\cos\theta)}{\sin^2\theta}\right)$$

since $\vec{E}$ is $\perp$ to $\hat{x}$ and $\vec{B}$ and $|\vec{E}| = c|\vec{B}|$ in the far field

$$\vec{E} = \frac{\mu_0 c}{4\pi}\,\mathrm{Re}\,\frac{e^{-i\omega t + ikx}}{x}(-2iI_0)(\hat{z})_\perp\left[\frac{\cos\frac{kd}{2} - \cos(\frac{kd}{2}\cos\theta)}{\sin^2\theta}\right]$$

and

$$\langle\vec{S}\rangle = \frac{\mu_0}{16\pi^2}\,\frac{1}{2}\cdot 4I_0^2 c\,\sin^2\theta\,\frac{1}{x^2}\left|\frac{\cos\frac{kd}{2} - \cos(\frac{kd}{2}\cos\theta)}{\sin^2\theta}\right|^2\hat{x}$$

so that, integrating over a piece of a sphere at a large distance away

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2} \cdot 2I_0^2 c \cdot \left| \frac{\cos\frac{kd}{2} - \cos\frac{kd}{2}\cos\theta}{\sin\theta} \right|^2$$

In the limit of a short antenna, the radiation should be mainly that of an oscillating electric dipole. Let's check ~~that~~ this is the case. For $kd \ll 1$

$$\frac{\cos\frac{kd}{2} - \cos\frac{kd}{2}\cos\theta}{\sin^2\theta} \rightarrow \frac{\left[1 - \frac{1}{2}\left(\frac{kd}{2}\right)^2 + \cdots\right] - \left[1 - \frac{1}{2}\left(\frac{kd}{2}\right)^2\cos^2\theta + \cdots\right]}{\sin^2\theta}$$

$$= -\frac{1}{8}(kd)^2 \, \frac{1-\cos^2\theta}{\sin^2\theta} = -\frac{1}{8}(kd)^2$$

so

$$\frac{dP}{d\Omega} = \frac{\mu_0}{32\pi^2 c} \, \omega^4 \, \sin^2\theta \cdot \left(\frac{I_0}{4c} d^2\right)^2$$

in this limit, the current at maximum amplitude is

$$I = I_0 \sin k\left(\frac{d}{2} - |z|\right) \cong I_0 \, k\left(\frac{d}{2} - |z|\right)$$

which implies a charge density

$$\rho_0 = \frac{1}{\omega}\frac{\partial}{\partial z} I = \frac{k}{\omega} I_0 \left(\frac{d}{2} - |z|\right) = \frac{I_0}{c}\left(\frac{d}{2} - |z|\right)$$

the dipole moment associated with this density is indeed

$$p_0 = 2 \cdot \frac{I_0}{c} \, \frac{1}{2}\left(\frac{d}{2}\right)^2 = \frac{I_0}{4c} d^2$$

When kd becomes large, the story acquires new features.
Some convenient cases to consider are the half-wave antennas:

$\frac{kd}{2} = m\frac{\pi}{2}$ with m odd. Then $\cos\frac{kd}{2} = 0$ and the integral $\mathcal{I}$ on p. 3 is proportional to

$$\mathcal{F} = \left[\frac{\cos\left(m\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\right]$$

Near $\theta = 0$, $\cos\theta \cong 1 - \frac{1}{2}\theta^2 + \dots$ so

$$\mathcal{F} \sim \frac{\cos m\pi/2\,(1-\frac{1}{2}\theta^2)}{\theta^2} \sim \frac{\cos\frac{m\pi}{2}\cos\frac{m\pi}{4}\theta^2 + \sin\frac{m\pi}{2}\sin\frac{m\pi}{4}\theta^2}{\theta^2}$$

$$\sim \frac{0 + (-1)^{(m-1)/2}\frac{m\pi}{4}\theta^2}{\theta^2}$$

$$\longrightarrow \text{constant.}$$

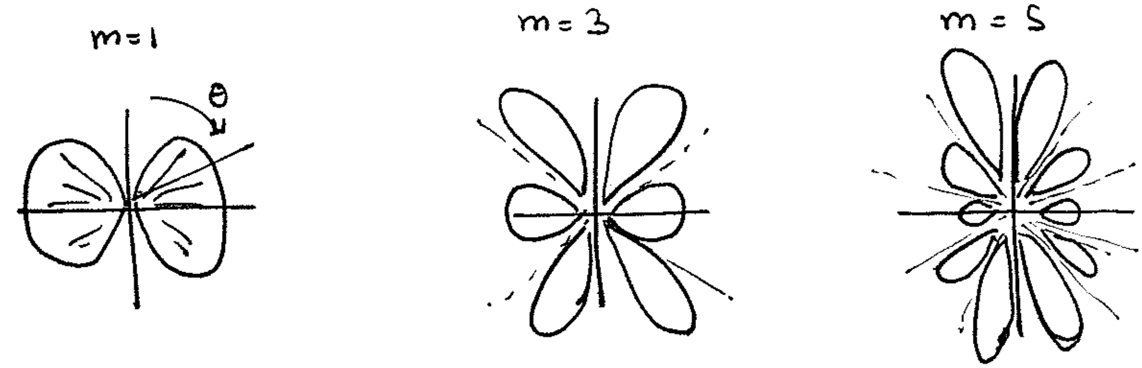
and since $|\hat{x}\times\hat{z}| = \sin\theta$ has a zero, the radiated power vanishes. The same conclusion holds at $\theta = \pi$. $\mathcal{F}$ also vanishes when

$$\cos\theta = \frac{n}{m} \qquad n = \text{odd}$$

and has maxima near points where

$$\cos\theta = \frac{n}{m} \qquad n = \text{even}.$$

Thus we find the patterns



(the maxima values are larger at θ close to $0, \pi$ because of the $(\sin\theta)^{-2}$ factor.) and, more generally

$$\frac{dP}{d\Omega} = \frac{\mu_0}{8\pi^2} I_0^2 c \left| \frac{\cos\left(m\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right|^2$$

The total power is obtained by integrating $\frac{dP}{d\Omega}$ over solid angle. I'll quote two illustrative cases.

short dipole antenna : $\quad P = \frac{\mu_0 c}{192\pi} I_0^2 (kd)^4$

m=1 half-wave : $\quad P = \frac{\mu_0}{8\pi} I_0^2 c \cdot (2.44)$

Recall that the Power dissipated by a resistance is

$$P = IV = I^2 R$$

For an alternating current, $\quad \langle P \rangle = \frac{1}{2} I_0^2 R$

So we can re express these formulae as expressions for the effective resistance of the antenna. We might call this the radiation resistance, R_{rad}, because the power is dissipated, not into heating the wire, but rather into radiation. To put this resistance into convenient units, rearrange

$$\mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0}} = 4\pi\times10^{-7}\ \frac{N}{A^2}\cdot 3.0\times10^{8}\,m/sec$$

$$= 377\ \frac{N\text{-}m/sec}{A^2}$$

the units are $\frac{J/sec}{A^2} = \frac{J/C}{A} = \Omega$! so

$$\mu_0 c = 377\ \Omega$$

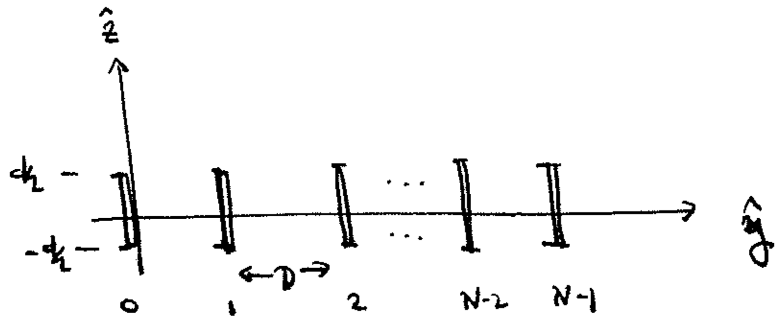
and, for the two cases above: $P = \frac{1}{2} R_{rad} I_0^2$

with

short dipole: $R_{rad} = 1.25\left(\frac{2\pi d}{\lambda}\right)^4\ \Omega$

m=1 half-wave: $R_{rad} = 73.2\ \Omega$

We can further shape the antenna radiation pattern by considering an array of antennas. Again, I'll analyze only the simplest case. Let's take an array of N half-wave antennas arranged along the $\hat{y}$ axis at a separation D.



and all driven in phase, so that $I = I_0 \sin k(\frac{d}{2} - |x|) \cos \omega t$ for each antenna.

Superposing the $\vec{A}$ fields generated by each antenna, we have

$$\vec{A}(t, \vec{x}) = \frac{\mu_0}{4\pi} \operatorname{Re} \frac{e^{-i\omega t + ikx}}{x} \cdot \sum_{j=0}^{N-1} \int_{-d/2}^{d/2} dz \; I_0 \sin k(\tfrac{d}{2} - z)\, \hat{z} \cdot e^{ik\hat{x}\cdot(z\hat{z} + jD\hat{y})}$$

$$= \frac{\mu_0}{4\pi} \operatorname{Re} \frac{e^{-i\omega t + ikx}}{x} \; \hat{z} \left(+\frac{2I_0}{k}\right) \left(\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}\right) \cdot \left(\sum_{j=0}^{N-1} e^{ikD\, \hat{x}\cdot\hat{y}\, \cdot j}\right)$$

We have already analyzed the various pieces of the first line of this expression. The new piece is.

$$\sum_{j=0}^{N-1} e^{ij\,kD\,\hat{x}\cdot\hat{y}} \qquad\qquad \alpha = kD\,\hat{x}\cdot\hat{y}$$

$$= 1 + e^{i\alpha} + e^{2i\alpha} + \cdots e^{(N-1)i\alpha}$$

$$= \frac{1 - e^{Ni\alpha}}{1 - e^{i\alpha}} = \frac{\sin\frac{N\alpha}{2}}{\sin\frac{\alpha}{2}}.$$

here $\alpha = kD\,\hat{x}\hat{y} = kD\sin\theta\cos\phi$. Then, following the steps above:

$$\vec{B} = \frac{\mu_0}{4\pi}\,\mathrm{Re}\,\frac{e^{-i\omega t + ikx}}{x}\,(2iI_0)(\hat{x}\times\hat{z})\,\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}\,\frac{\sin\frac{N\alpha}{2}}{\sin\frac{\alpha}{2}}$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2}\,I_0^2\left|\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right|^2\left|\frac{\sin N\alpha/2}{\sin\alpha/2}\right|^2$$

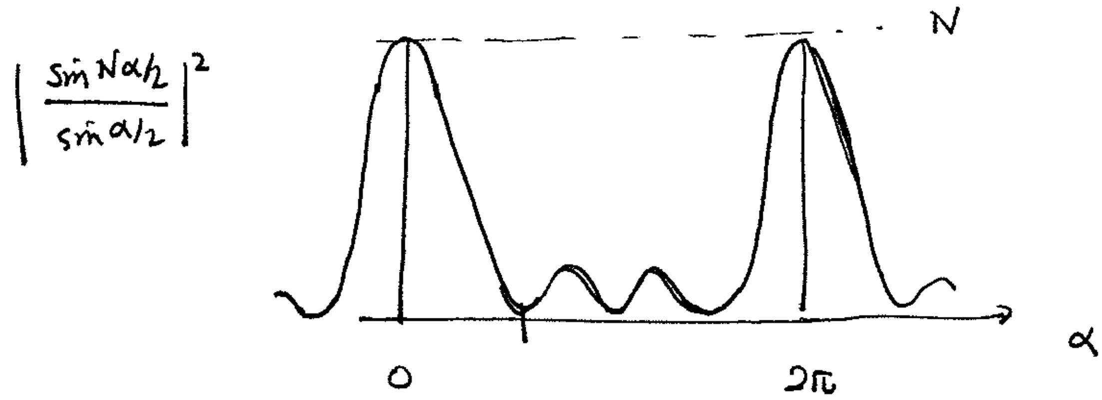
The function $\frac{\sin N\alpha/2}{\sin\alpha/2}$ is an important one that is worth some attention to understand. Near $\alpha = 0$

$$\frac{\sin N\alpha/2}{\sin\alpha/2} \sim \frac{N\alpha/2}{\alpha/2} \sim N$$

The function is periodic, so there are identical maxima at

$$\alpha = 2\pi n$$

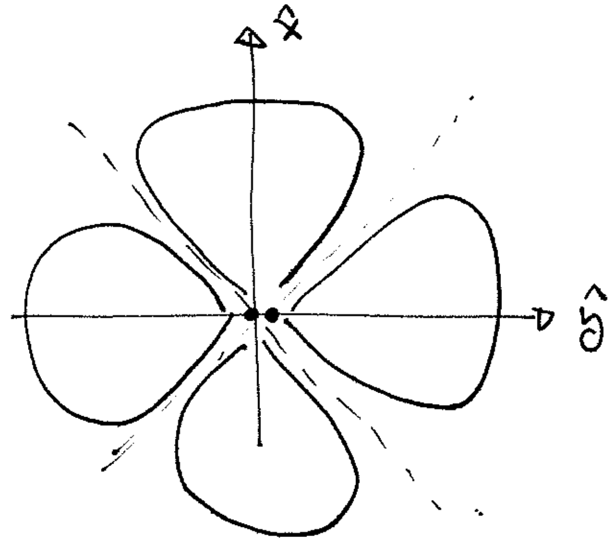
At $\alpha = \frac{2\pi}{N}(m+\frac{1}{2})$ the sin function in the numerator equals ± 1, so there are lower local maxima near these points. In all:



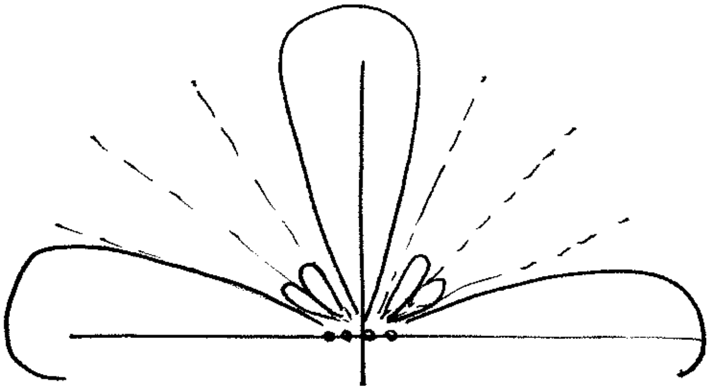
the first zero is at $\alpha = \frac{2\pi}{N}$, the first max after this is near $\frac{\frac{5}{2}\pi}{N}$.

Replacing $\alpha = kD\sin\theta\sin\phi = 2\pi\frac{D}{\lambda}\sin\theta\sin\phi$, we see that this generates a radiation pattern with maxima and minima as a function of ϕ. For $\theta = \pi/2$, $\sin\theta = 1$, and the illustrative case $D/\lambda = 1$:

N=2

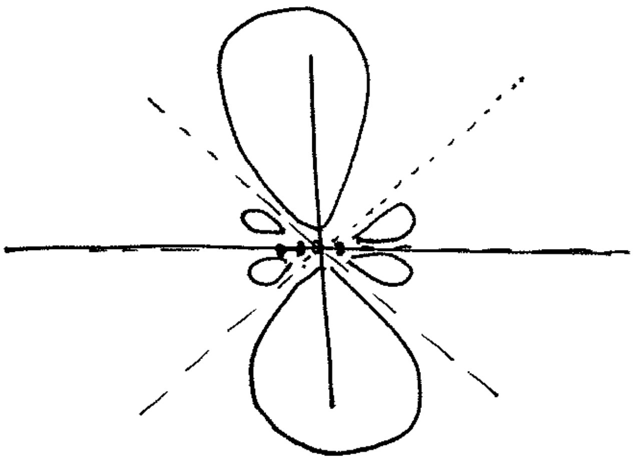


N = 4



for $D/\lambda < 1$, we never reach the second maximum and so the only major peak of the radiation pattern is along the $\hat{x}$ axis

e.g. $D/\lambda = \frac{1}{2}$ $N = 4$



We will encounter this pattern again later in the course when we study diffraction.