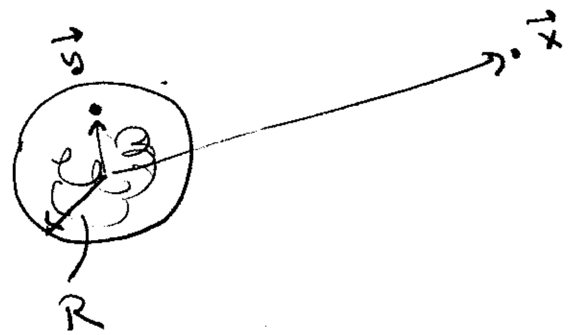
Multipole Radiation

April 25

To make a more general analysis of radiation from accelerated charge, I would first like to step back to a situation that we analyzed for static charges in 120. Consider a set of charges and currents localized to the interior of a sphere of radius R:



I would like to analyze the fields at a point $\vec{x}$ s.t. $|\vec{x}| > R$. In 120, we noticed ~~that~~, in this setup, we can expand

$$\frac{1}{|\vec{x}-\vec{y}|} = \frac{1}{x} + \frac{\vec{y}\cdot\hat{x}}{x^2} + \cdots$$

and then interpret each successive term as the potential of a higher moment or "multipole" of the charge and current distribution. I would like to carry out this same analysis for dynamic fields. For simplicity, I will consider charges and currents that oscillate with frequency ω:

$$\rho(t,\vec{y}) = \mathrm{Re}\ \rho_0(\vec{y})\ e^{-i\omega t}$$

$$\vec{j}(t,\vec{y}) = \mathrm{Re}\ \vec{j}_0(\vec{y})\ e^{-i\omega t}$$

[Note that current conservation $\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0$ implies

$$-i\omega\rho_0(\vec{y}) + \vec{\nabla}\cdot\vec{j}_0(\vec{y}) = 0 .\ ]$$

In a real situation, the charges and currents may have more complicated time-dependence, but we can consider the most general function of time as a sum of Fourier components with definite ω.

The ρ and $\vec{j}$ above lead to the potentials

$$\phi(t,\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{t} dt_y \int d^3y\ \frac{1}{|\vec{x}-\vec{y}|}\ \mathrm{Re}\ \rho_0(\vec{y})\, e^{-i\omega t_y}$$

$$\cdot\ \delta\left(t - t_y - \frac{|\vec{x}-\vec{y}|}{c}\right)$$

and similarly for $\vec{A}$. We can then integrate over the δ-function, replacing t_y by $t - \frac{|\vec{x}-\vec{y}|}{c}$:

$$\phi(t,\vec{x}) = \frac{1}{4\pi\epsilon_0}\ \mathrm{Re}\ e^{-i\omega t}\int d^3y\ \frac{\rho_0(\vec{y})}{|\vec{x}-\vec{y}|}\ e^{i\frac{\omega}{c}|\vec{x}-\vec{y}|}$$

ω is the frequency of the radiation that will be emitted, and $k = \omega/c$ is the corresponding wavenumber.

Looking at this expression, it is clear that $\phi(t,\vec{x})$ has different behaviors in different regions of $|\vec{x}|$. Let me specialize to the case

$$\frac{\omega}{c} R = kR \ll 1 \quad \text{or} \quad R/\lambda \ll 1$$

where $\lambda = \frac{2\pi}{k}$ is the wavelength of the emitted radiation. This condition is also

$$\omega = \frac{2\pi}{T} \qquad \frac{R}{c} = t_\ell \text{, the time for light to travel across } R$$

$$\text{so} \quad kR \ll 1 \quad \Rightarrow \quad t_\ell / T \ll 1$$

and so this is also the limit of nonrelativistic motion of the charges.

Now consider the above expression for $\phi(t,\vec{x})$ for

$$R < |\vec{x}| \ll \lambda$$

In this limit, we can ignore $\frac{\omega}{c}|x-y| = k|\vec{x}-\vec{y}|$ and find

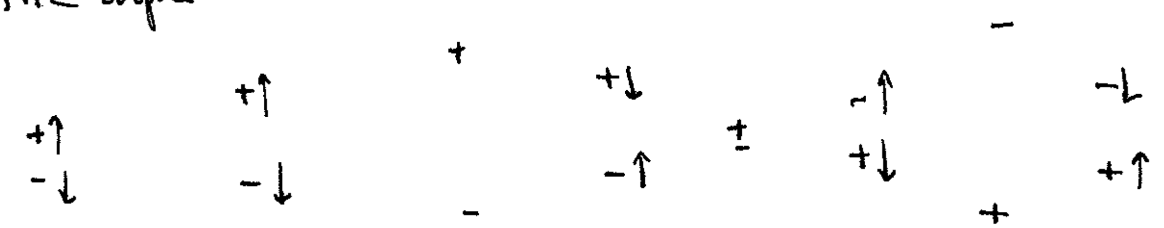
$$\phi(t,\vec{x}) \cong \text{Re}\, e^{-i\omega t} \frac{1}{4\pi\epsilon_0} \int d^3y \frac{\rho_0(\vec{y})}{|\vec{x}-\vec{y}|} \qquad |x| \ll \lambda$$

$$A(t,\vec{x}) \cong \text{Re}\, e^{-i\omega t} \frac{1}{4\pi\epsilon_0 c^2} \int d^3y \frac{\vec{J}_0(\vec{y})}{|\vec{x}-\vec{y}|}$$

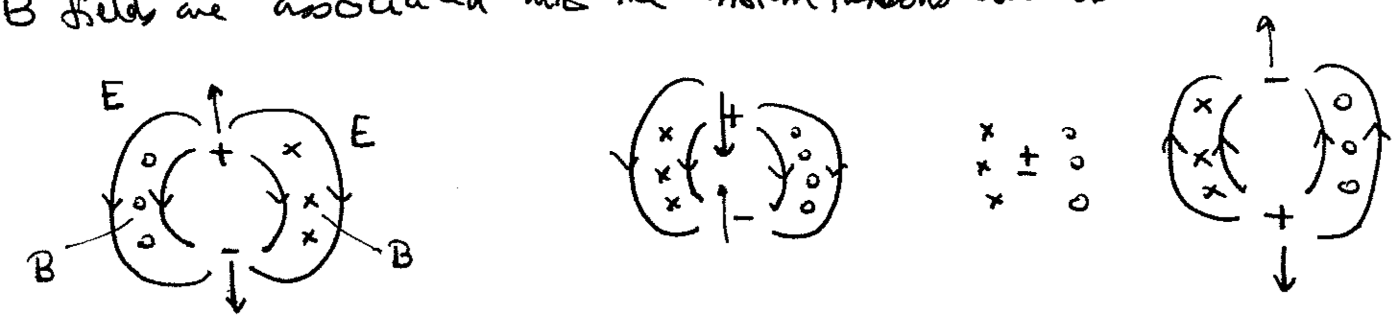
These are just the formulae for the static scalar and vector potentials, for the charges and currents and oscillating at the frequency ω. This region is called the "near field". Radiation proper will appear in the opposite limit,

$$|\vec{x}| \gg \lambda \gg R$$

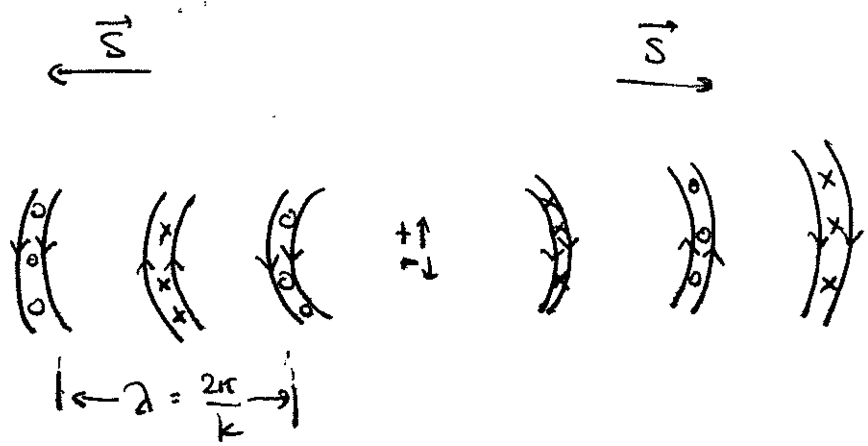
the "far field". To see how radiation emerges from the near field region, consider the particular case of an oscillating electric dipole:



Let's draw the associated quasi-static $\vec{E}$ and $\vec{B}$ fields (the $\vec{B}$ fields are associated with the instantaneous currents):



Notice that in the first and last pictures, the Poynting vector $\vec{S}$ points outward — energy is being transferred to infinity. In the large, we have expanding shells of energy, each moving outward at the speed of light and remembering the configuration of the dipole at the time they were created. So,



Let's now try to compute the radiation pattern more explicitly.

For this, we should make the approximations appropriate to the far field

$$\frac{\omega}{c}|\vec{x}-\vec{y}| = k\left[x^2 - 2\vec{x}\cdot\vec{y} + (\vec{y})^2\right]^{1/2}$$

$$= k(x)\left(1 - \frac{\vec{x}\cdot\vec{y}}{x^2} + \frac{1}{2}\frac{y^2}{x^2} - \frac{1}{8}\frac{(2\vec{x}\cdot\vec{y}+\cdots)^2}{x^4} + \cdots\right)$$

$$= kx - k\vec{y}\cdot\hat{x} + O\left(\frac{ky^2}{x}\right)$$

and note that $ky < kR \ll 1$, $\frac{y}{x} \ll 1$

so the terms dropped are negligible in the far field.

$$\phi(t,\vec{x}) \cong \frac{1}{4\pi\epsilon_0}\,\mathrm{Re}\, e^{-i\omega t} e^{ik|\vec{x}|}\int d^3y\, \frac{\rho_0(y)}{|x-y|}\, e^{-ik\hat{x}\cdot\vec{y}}$$

$$\vec{A}(t,\vec{x}) \cong \frac{1}{4\pi\epsilon_0 c^2}\,\mathrm{Re}\, e^{-i\omega t} e^{ik|\vec{x}|}\int d^3y\, \frac{\vec{J}_0(y)}{|x-y|}\, e^{-ik\hat{x}\cdot\vec{y}}$$

the first approximation to the scalar potential is

$$\Phi(t,\vec{x}) \cong \frac{1}{4\pi\epsilon_0} \mathrm{Re}\, e^{-i\omega t + ik|\vec{x}|} \int d^3y \, \frac{\rho_0(y)}{|\vec{x}|}$$

But $\int d^3y \, \rho_0(\vec{y})$ is the Fourier component of the total charge in the sphere of radius R. Since

$$Q = \int d^3y \, \rho(t,\vec{y}) = \underline{\text{constant}}$$

the Fourier component $\int d^3y \, \rho_0(\vec{y}) \, e^{-i\omega t}$ must be $= 0$!

The next approximation is obtained from expanding

$$\frac{1}{|\vec{x}-\vec{y}|} \cong \frac{1}{|\vec{x}|} + \frac{\vec{y}\cdot\hat{x}}{|\vec{x}|^2} + \cdots$$

$$e^{-ik\hat{x}\cdot\vec{y}} \cong 1 - ik\,\hat{x}\cdot\vec{y}$$

The first approximation is smaller by a factor y/x. The second is smaller by a factor ky. In the near field

$$\frac{y}{x} \gg ky$$

and we obtain a quasistatic dipole field. In the far field

$$ky \gg \frac{y}{x}$$

and so

$$\Phi(t,\vec{x}) \cong \frac{1}{4\pi\epsilon_0} \mathrm{Re}\, e^{-i\omega t + ik|\vec{x}|} \int d^3y \, \frac{\rho_0(\vec{y})}{x} (-ik\,\hat{x}\cdot\vec{y})$$

We recognize

$$\int d^3y \, \vec{y} \, \rho_0(\vec{y}) = \vec{p}_0$$

the Fourier component of the electric dipole moment,

$$\vec{p}(t) = \mathrm{Re} \, e^{-i\omega t} \vec{p}_0(t)$$

Then

$$\phi(t,\vec{x}) = \frac{1}{4\pi\epsilon_0} \mathrm{Re} \frac{e^{-i\omega t + ikx}}{x} \cdot (-ik\hat{x}\cdot\vec{p}_0)$$

The expression

$$\frac{e^{-i\omega t + ikx}}{x}$$

is an outward-moving scalar spherical wave, so we are going into the right territory.

For the $\vec{A}$ field, we can take the first approximation

$$\vec{A}(t,\vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \mathrm{Re} \, e^{-i\omega t + ikx} \int d^3y \, \frac{\vec{J}_0(\vec{y})}{x}$$

Since $\vec{J}(\vec{y})$ is confined to a sphere of radius R

$$0 = \int d^3y \, \vec{\nabla}(y^i \vec{J}_0) = \int d^3y \, \{ J_0^i(\vec{y}) + y^i (\vec{\nabla}\cdot\vec{J}_0) \}$$

$$= \int d^3y \, \{ J_0^i(\vec{y}) + y^i \cdot i\omega \rho_0(\vec{y}) \}$$

using the equation of current conservation on p. 2

Then

$$\vec{A}(t,\vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \, Re \, \frac{e^{-i\omega t + ikx}}{x} \int d^3y \, (-i\omega \, \rho_0(\vec{y})) \, \vec{y}$$

$$= \frac{1}{4\pi\epsilon_0 c^2} \, Re \, \frac{e^{-i\omega t + ikx}}{x} \, (-i\omega \vec{p}_0)$$

again

$$\phi(t,\vec{x}) = \frac{1}{4\pi\epsilon_0 c} \, Re \, (-i\omega) \, \frac{e^{-i\omega t + ikx}}{x} \, (\hat{x}\cdot\vec{p}_0)$$

$$\vec{A}(t,\vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \, Re \, (-i\omega) \, \frac{e^{-i\omega t + ikx}}{x} \, (\vec{p}_0)$$

To compute the fields, we have to take

$$\vec{\nabla}\left(\frac{e^{ikx}}{x}\right) = \left[\left(ik\frac{\vec{x}}{x}\right)\frac{1}{x} - \frac{\vec{x}}{x^3}\right] e^{ikx}$$

$$= \left(\frac{ik\hat{x}}{x} - \frac{\hat{x}}{x^2}\right) e^{ikx}$$

in the far field, the first term dominates:

$$\vec{\nabla}\left(\frac{e^{ikx}}{x}\right) \cong ik\hat{x} \, \frac{e^{ikx}}{x} \qquad x \gg \lambda$$

Then:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$$

$$= \frac{1}{4\pi\epsilon_0 c} \, Re \, (-i\omega) \, \frac{e^{-i\omega t + ikx}}{x} \left\{-ik\,\hat{x}\;\hat{x}\cdot\vec{p}_0 + i\frac{\omega}{c}\vec{p}_0\right\}$$

so

$$\vec{E} = \frac{1}{4\pi\epsilon_0 c^2} \, \mathrm{Re} \, \omega^2 \frac{e^{-i\omega t + ikx}}{x} \left(\vec{p}_0 - \hat{x}\,\hat{x}\cdot\vec{p}_0 \right)$$

$$\vec{B} = \frac{1}{4\pi\epsilon_0 c^2} \, \mathrm{Re} \, \frac{\omega^2}{c} \frac{e^{-i\omega t + ikx}}{x} \, \hat{x} \times \vec{p}_0$$

Very nicely, $\vec{E}$ is proportional to the component of $\vec{p}_0$ perpendicular to $\vec{x}$, and $\vec{B} = \frac{1}{c}\hat{x}\times\vec{E}$. These are just the properties we found for radiation from isolated charges. The Poynting vector is

$$\vec{S} = \frac{1}{\mu_0} \langle \vec{E} \times \vec{B} \rangle$$

$$= \frac{1}{2} \cdot \frac{\mu_0}{16\pi^2} \frac{\omega^4}{c} \frac{1}{x^2} \cdot \left| (\vec{p}_0)_\perp \right|^2 \hat{x}$$

If $\vec{p}_0$ is directed along the $\hat{z}$ axis.

$$= \frac{\mu_0}{32\pi^2} \frac{\omega^4}{c} \frac{1}{x^2} p_0^2 \sin^2\theta \; \hat{x}$$

The radiated power is

$$P = \int d^2x \; \hat{n}\cdot\vec{S} = \int d\Omega \, x^2 \; \hat{x}\cdot\vec{S}$$

So

$$\frac{dP}{d\Omega} = \frac{\mu_0}{32\pi^2}\frac{\omega^4}{c} p_0^2 \sin^2\theta$$

$$P = \frac{\mu_0}{12\pi}\frac{\omega^4}{c} p_0^2$$

To relate this to our previous calculation, consider a charge q executing simple harmonic motion:

$$z(t) = A\cos\omega t$$

The monopole moment — the total charge — is q, a constant.
The dipole moment is

$$\vec{p}(t) = qA\cos\omega t\,\hat{z}$$

so $p_0 = qA\hat{z}$ and

$$P = \frac{\mu_0 q^2}{12\pi c}|\omega^2 A|^2$$

The acceleration of the charge is

$$\vec{a} = -qA\omega^2\cos\omega t\,\hat{z}$$

So $$\langle|\vec{a}|^2\rangle = \frac{q^2}{2}(A\omega^2)^2$$

then

$$P = \frac{\mu_0 q^2}{6\pi c} \langle |\vec{a}|^2 \rangle$$

just as we would have predicted from Larmor's formula. The angular distribution of the power is

$$\frac{dP}{d\Omega} \sim \sin^2\theta$$

about the axis of $\vec{a}$, just as in the previous case. However, our new relations are more general. They apply to any small oscillating system of charges and currents, to the first approximation.

Let's continue on and study the next approximation. Following the estimates above, I will keep the leading terms in the far field region. For ϕ

$$\phi_{(2)}(t,\vec{x}) = \frac{1}{4\pi\epsilon_0} \mathrm{Re}\, \frac{e^{-i\omega t + ikx}}{x} \int d^3y\; \rho_0(\vec{y}) \left(\frac{-ik\hat{x}\cdot\vec{y}}{2}\right)^2$$

and for $\vec{A}$

$$\vec{A}_{(2)}(t,\vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \mathrm{Re}\, \frac{e^{-i\omega t + ikx}}{x} \int d^3y\; \vec{J}_0(\vec{y}) \left(-ik\,\hat{x}\cdot\vec{y}\right)$$

To simplify these expressions, let's begin by working on the formula for $\vec{A}_{(2)}$

$$(\hat{x}\cdot\vec{y})\,\vec{J}_0(\vec{y})$$

$$= \tfrac{1}{2}\left[(\hat{x}\cdot\vec{y})\vec{J}_0(y) + (\hat{x}\cdot\vec{J}_0)\vec{y}\right] + \tfrac{1}{2}\left[(\hat{x}\cdot\vec{y})\vec{J}_0(y) - (\hat{x}\cdot\vec{J}_0)\vec{y}\right]$$

$$= \tfrac{1}{2}\left[(\hat{x}\cdot\vec{y})\vec{J}_0 + (\hat{x}\cdot\vec{J}_0)\vec{y}\right] + \tfrac{1}{2}\,\hat{x}\times(\vec{J}_0\times\vec{y})$$

Taking for the moment only the second term, we find

$$\vec{A}_{(2),M} = \frac{1}{4\pi\epsilon_0 c^2}\,\mathrm{Re}\,\frac{e^{-i\omega t+ikx}}{x}\,(+ik\hat{x})\times\int d^3y\,\frac{\vec{y}\times\vec{J}_0}{2}$$

We recognize ~~the integral~~ as the Fourier component of the magnetic dipole moment of the current distribution

$$\vec{m} = \int d^3y\;\tfrac{1}{2}\,\vec{y}\times\vec{J}$$

Thus, we can isolate the contribution

$$\phi_{(2),M} = 0$$

$$A_{(2),M} = \frac{\mu_0}{4\pi}\,\mathrm{Re}\,\frac{e^{-i\omega t+ikx}}{x}\;ik\,(\hat{x}\times\vec{m}_0)$$

These terms lead to the fields in the far field region:

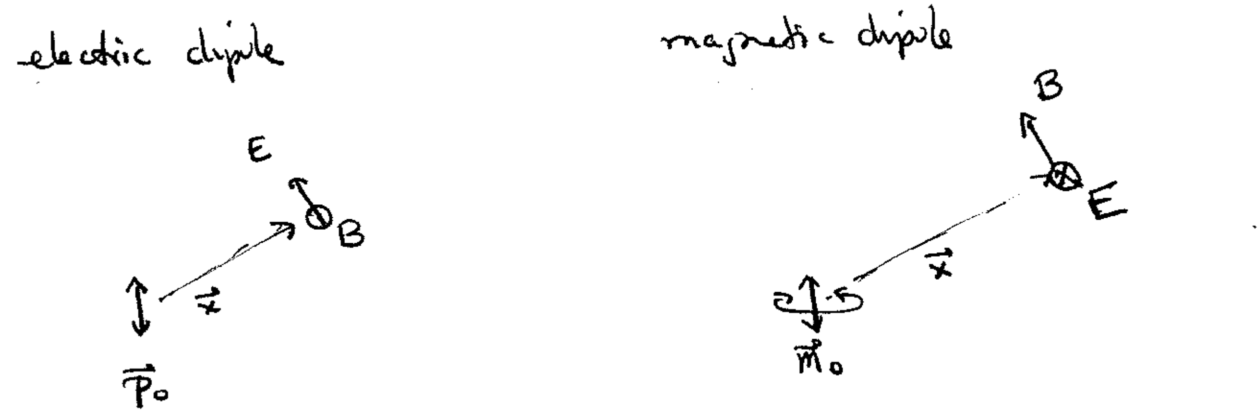
$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A} = \frac{\mu_0}{4\pi}\,\mathrm{Re}\,\frac{e^{-i\omega t+ikx}}{x}\left(-\frac{\omega^2}{c}\right)\hat{x}\times\vec{m}_0$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\,\mathrm{Re}\,\frac{e^{-i\omega t+ikx}}{x}\left(-\frac{\omega^2}{c^2}\right)\hat{x}\times(\hat{x}\times\vec{m}_0)$$

As required, these fields fall off as $\frac{1}{x}$ and satisfy

$$\vec{E} \perp \hat{x}\ , \quad \vec{B} \perp \hat{x}\ , \quad \vec{B} \perp \vec{E}$$

The fields depend on the component of $\vec{m}_0$ perpendicular to $\hat{x}$. However, the relation of $\vec{E}$ and $\vec{B}$ to the motion of charges is reversed from the electric dipole case:



We can compute the power radiated by an oscillating magnetic dipole as on p. 9

$$\vec{S} = \frac{1}{\mu_0}\langle \vec{E}\times\vec{B}\rangle = \frac{\mu_0}{16\pi^2}\ \frac{1}{x^2}\ \frac{\omega^4}{c^3}\ \frac{1}{2}\ |(\vec{m}_0)_\perp|^2\ \hat{x}$$

$$\frac{dP}{d\Omega} = \frac{\mu_0}{32\pi^2}\ \frac{\omega^4}{c^3}\ m_0^2 \sin^2\theta$$

$$P = \frac{\mu_0}{12\pi}\ \frac{\omega^4}{c^3}\ |\vec{m}_0|^2$$

Notice that the units of $\vec{m}_0$ as

$$C - m - m/sec$$

so the dimensions are correct with respect to p.10. The power radiated by a magnetic dipole is smaller that that radiated by the electric dipole by a factor

$$\left(\frac{v}{c}\right)^2$$

where v is the velocity with which the charges move.

Now return to the first term at the top of p.12. Using

$$\nabla_y^i \left[y^j (\hat{x}\cdot\vec{y})\, j_0^k \right]$$

$$= \left[\delta^{ij}\, \hat{x}\cdot\vec{y}\, j_0^k + y^j \hat{x}^i j_0^k + y^j (\hat{x}\cdot\vec{y})\, \nabla^i j_0^k \right]$$

and setting $i=k$, we have

$$\vec{\nabla}_y \cdot \left[y^j (\hat{x}\cdot\vec{y})\, \vec{j}_0 \right]$$

$$= \left[(\hat{x}\cdot\vec{y})\, j_0^j + (\hat{x}\cdot\vec{j}_0)\, y^j + y^j (\hat{x}\cdot\vec{y})\, \vec{\nabla}\cdot\vec{j}_0 \right]$$

$$= \left[(\hat{x}\cdot\vec{y})\, j_0^j + (\hat{x}\cdot\vec{j}_0)\, y^j + y^j (\hat{x}\cdot\vec{y})\, i\omega\rho_0 \right]$$

using the equation of current conservation. Now, since the charge distribution is localized

$$\int d^3y \; \vec{\nabla}_y \left[y^j (\hat{x}\cdot\vec{y}) \vec{J}_0(\vec{y}) \right] = 0$$

this implies

$$\int d^3y \; \tfrac{1}{2}\left[(\hat{x}\cdot\vec{y})\vec{J}_0 + (\hat{x}\cdot\vec{J}_0)\vec{y}\right]$$

$$= \int d^3y \; \tfrac{1}{2}(-i\omega)\, \rho_0(y)\; \vec{y}\; \vec{y}\cdot\hat{x}$$

so we find for the remaining term on p. 12

$$\phi_{(2),Q} = \frac{1}{4\pi\epsilon_0}\, \mathrm{Re}\, \frac{e^{-i\omega t + ikx}}{x} (-k^2) \int d^3y \; \rho_0(\vec{y}) \tfrac{1}{2}(\hat{x}\cdot\vec{y})^2$$

$$\vec{A}_{(2),Q} = \frac{1}{4\pi\epsilon_0 c^2}\, \mathrm{Re}\, \frac{e^{-i\omega t + ikx}}{x} \cdot (-k\omega)\tfrac{1}{2}\int d^3y \; \rho_0(\vec{y})\; \vec{y}(\vec{y}\cdot\hat{x})$$

The corresponding $\vec{E}$ field is $\quad (\omega = ck)$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$$

$$= \frac{1}{4\pi\epsilon_0}\, \mathrm{Re}\, \frac{e^{-i\omega t + ikx}}{x} (-ik^3)$$

$$\cdot \tfrac{1}{2}\int d^3y \; \rho_0(y) \left[\vec{y}(\vec{y}\cdot\hat{x}) - \hat{x}\; \hat{x}\cdot\vec{y}\; \vec{y}\cdot\hat{x} \right]$$

The integral can be related to the electric quadrupole moment of the charge distribution

$$Q^{ij} = \int d^3y \; \rho(\vec{y}) \left[3\, y^i y^j - \delta^{ij} y^2 \right]$$

Replacing ρ by $\rho_0(\vec{y})$, we obtain the Fourier component of Q^{ij}. Then

$$Q_0^{ij}\, \hat{x}^j = \int d^3y \; \rho_0(\vec{y}) \left[3\, y^i\, \vec{y}\cdot\hat{x} - (\hat{x})^i y^2 \right]$$

This is a vector. Project it perpendicular to $\hat{x}$

$$(\overrightarrow{Q_0 \cdot x})_\perp = (\delta^{ik} - \hat{x}^i \hat{x}^k)\; Q_0^{kj}\, \hat{x}^j$$

$$= \int d^3y \; \rho_0(\vec{y}) \Big\{ 3\, y^i\, \vec{y}\cdot\vec{x} - 3\, \hat{x}^i (\hat{x}\cdot\vec{y})\, \vec{y}\cdot\hat{x}$$

$$- \hat{x}^i y^2 + \hat{x}^i \underbrace{\hat{x}\cdot\hat{x}}_{1}\, y^2 \Big\}$$

$$= 3 \int d^3y \; \rho_0(\vec{y}) \left[y^i\, \vec{y}\cdot\hat{x} - \hat{x}^i (\hat{x}\cdot\vec{y})^2 \right]$$

so we recognize

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \,\mathrm{Re}\, \frac{e^{-i\omega t + ikx}}{x} \,(-ik^3)\, \frac{1}{6} \left(\overleftrightarrow{Q}_0 \cdot \vec{x} \right)_\perp$$

similarly,

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{4\pi\epsilon_0 c}\, \mathrm{Re}\, \frac{e^{-i\omega t + ikx}}{x}\, (-ik^3)$$

$$\cdot \frac{1}{2}\int d^3y\; \rho_0(\vec{y})\;\; \hat{x}\times(\vec{y}\;\vec{y}\cdot\hat{x})$$

$$= \frac{1}{4\pi\epsilon_0 c}\, \mathrm{Re}\, \frac{e^{-i\omega t + ikx}}{x}\, (-ik^3)\;\frac{1}{6}\,\hat{x}\times(\overleftrightarrow{Q_0}\cdot\hat{x})$$

again we have the conditions for radiation:

$$\vec{E}\perp\hat{x}\;, \quad \vec{B}\perp\hat{x} \quad \vec{B}\perp\vec{E} \quad |\vec{E}|, |\vec{B}| \sim \frac{1}{x}$$

comparing to the dipole fields on p. 8-9, the quadrupole contributions are smaller by

$$\frac{k^3 Q_0}{k^2 p_0} \sim kR$$

Let's now compute the Poynting vector and the angular distribution of radiation

$$\langle\vec{S}\rangle = \frac{1}{\mu_0}\langle\vec{E}\times\vec{B}\rangle$$

$$= \frac{\mu_0}{(4\pi)^2}\cdot\left(\frac{1}{6}\right)^2 \frac{\omega^6}{c^3}\;\frac{1}{x^2}\cdot\frac{1}{2}\left|\overleftrightarrow{Q_0}\hat{x}\right|_{\perp}^2$$

$$= \frac{\mu_0}{1152\pi^2}\,\frac{\omega^6}{c^3}\,\frac{1}{x^2}\,\left|\overleftrightarrow{Q_0}\cdot\hat{x}\right|^2$$

so

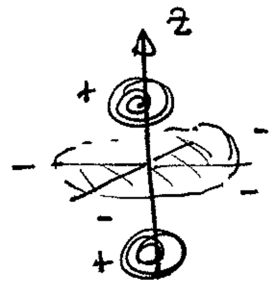
$$\frac{dP}{d\Omega} = \frac{\mu_0}{1152\pi^2} \frac{\omega^6}{c^3} \left|(Q_0 \hat{x})_\perp\right|^2$$

This formula is a little obscure, but we can make it clearer by evaluating it for some special cases. First recall that, since Q_0^{ij} is a symmetric matrix, we can find a coordinate system in which it takes the form

$$Q_0^{ij} = \begin{pmatrix} Q_{10} & & \\ & Q_{20} & \\ & & Q_{30} \end{pmatrix}$$

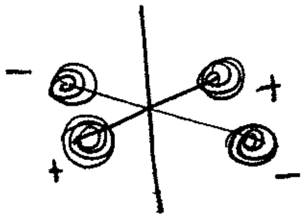
Since $Q_0^{ii} = 0$ or $\mathrm{tr}\, Q_0 = 0$, $\quad Q_{10} + Q_{20} + Q_{30} = 0$

For a cylindrically symmetric quadrupole distribution



$$Q_{10} = Q_{20} = -\frac{1}{2} Q_{30}$$

and the 3 axis lines up with $\hat{z}$. Another possible quadrupole distribution is:



for which $Q_{10} = -Q_{20}$, $Q_{30} = 0$

Now

$$\hat{x} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$Q_0\hat{x} = (Q_{10}\sin\theta\cos\phi,\ Q_{20}\sin\theta\sin\phi,\ Q_{30}\cos\theta)$$

$$(\hat{x}Q_0\hat{x}) = Q_{10}\sin^2\theta\cos^2\phi + Q_{20}\sin^2\theta\sin^2\phi + Q_{30}\cos^2\theta$$

and

$$(Q_0\hat{x})_\perp = Q_0\hat{x} - \hat{x}(\hat{x}Q_0\hat{x})$$

For the cylindrically symmetric case:

$$\begin{aligned}\hat{x}Q_0\hat{x} &= Q_{10}\sin^2\theta + Q_{30}\cos^2\theta \\ &= -\tfrac{1}{2}Q_{30}(1-\cos^2\theta) + Q_{30}\cos^2\theta \\ &= -\tfrac{1}{2}Q_{30}(1-3\cos^2\theta)\end{aligned}$$

$$Q_0\hat{x} = \left(-\tfrac{1}{2}Q_{30}\sin\theta\cos\phi,\ -\tfrac{1}{2}Q_{30}\sin\theta\sin\phi,\ Q_{30}\cos\theta\right)$$

$$\begin{aligned}(Q_0\hat{x})_\perp = \Big(&\sin\theta\cos\phi\left(-\tfrac{1}{2}Q_{30} + \tfrac{1}{2}Q_{30}(1-3\cos^2\theta)\right), \\ &\sin\theta\sin\phi\left(-\tfrac{1}{2}Q_{30} + \tfrac{1}{2}Q_{30}(1-3\cos^2\theta)\right), \\ &\cos\theta\left(Q_{30} + \tfrac{1}{2}Q_{30}(1-3\cos^2\theta)\right)\Big)\end{aligned}$$

$$= \Big(\sin\theta \cos\phi \left(-\tfrac{3}{2}\cos^2\theta\right),$$

$$\sin\theta \sin\phi \left(-\tfrac{3}{2}\cos^2\theta\right),$$

$$\cos\theta \ \tfrac{3}{2}\left(1-\cos^2\theta\right) \Big) \ Q_{30}$$

or

$$(Q_0 \hat{x})_\perp = Q_{30} \left(-\tfrac{3}{2}\right) \left(\sin\theta \cos\phi \cos^2\theta, \ \sin\theta \sin\phi \cos^2\theta, -\cos\theta \sin^2\theta \right)$$

$$\left(|Q_0 \hat{x}|_\perp\right)^2 = |Q_{30}|^2 \frac{9}{4} \left(\sin^2\theta \cos^2\phi \cos^4\theta + \sin^2\theta \sin^2\phi \cos^4\theta \right.$$

$$\left. + \cos^2\theta \sin^4\theta \right)$$

$$= \frac{9}{4} |Q_{30}|^2 \left(\sin^2\theta \cos^4\theta + \cos^2\theta \sin^4\theta \right)$$

$$= \frac{9}{4} |Q_{30}|^2 \ \sin^2\theta \cos^2\theta \left(\cos^2\theta + \sin^2\theta \right)$$

$$= \frac{9}{4} |Q_{30}|^2 \ \sin^2\theta \cos^2\theta$$

For the case arrayed on the x, y axes.

$$Q_0 \hat{x} = \left(Q_{10} \sin\theta \cos\phi, \ -Q_{10} \sin\theta \sin\phi, 0 \right)$$

$$\hat{x} \, Q_0 \hat{x} = Q_{10} \ \sin^2\theta \left(\cos^2\phi - \sin^2\phi \right) = Q_{10} \ \sin^2\theta \ \cos 2\phi$$

$$(Q_0 \hat{x})_\perp = \Big(Q_{10} \ \sin\theta \cos\phi \ \left(1 - \sin^2\theta \cos 2\phi \right),$$

$$Q_{10} \sin\theta \sin\phi \left(-1 - \sin^2\theta \cos 2\phi \right),$$

$$-\cos\theta \ Q_{10} \ \sin^2\theta \cos 2\phi \Big)$$

so that

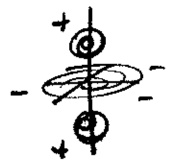
$$\left| \left[Q_0 \hat{x} \right]_\perp \right|^2 = |Q_{10}|^2 \Big(\sin^2\theta \cos^2\phi \, (1 - \sin^2\theta \cos 2\phi)^2$$
$$+ \sin^2\theta \sin^2\phi \, (1 + \sin^2\theta \cos 2\phi)^2$$
$$+ \cos^2\theta \sin^4\theta \cos^2 2\phi \Big)$$

$$= |Q_{10}|^2 \Big(\sin^2\theta + 2 \sin^4\theta \cos 2\phi \, (-\cos^2\phi + \sin^2\phi)$$
$$+ \sin^2\theta \, (\sin^4\theta \cos^2 2\phi)$$
$$+ \cos^2\theta \, (\sin^4\theta \cos^2 2\phi) \Big)$$

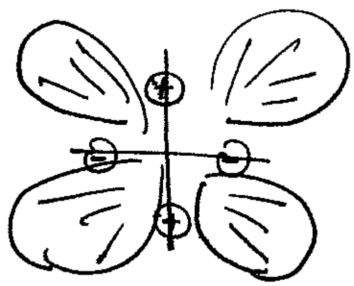
$$= |Q_{10}|^2 \left(\sin^2\theta + (-2) \sin^4\theta \cos^2 2\phi + \sin^4\theta \cos^2 2\phi \right)$$

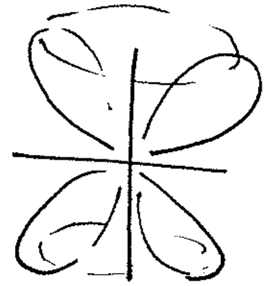
$$= |Q_{10}|^2 \left(\sin^2\theta - \sin^4\theta \cos^2 2\phi \right)$$

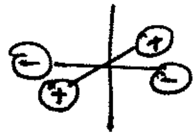
so, finally:



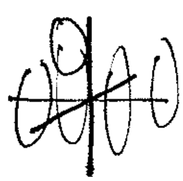
$$\frac{dP}{d\Omega} = \frac{\mu_0}{1152\pi^2} \frac{\omega^6}{c^3} \cdot |Q_{30}|^2 \, \frac{9}{4} \sin^2\theta \cos^2\theta$$

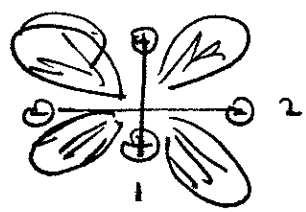






$$\frac{dP}{d\Omega} = \frac{\mu_0}{1152\pi^2} \frac{\omega^6}{c^3} \, |Q_{10}|^2 \sin^2\theta \left(1 - \sin^2\theta \cos^2 2\phi \right)$$





Integrate over $d\Omega$

$$\int d\cos\theta \, d\phi \ \sin^2\theta \cos^2\theta = 2\pi \int d\cos\theta \ \cos^2\theta\,(1-\cos^2\theta)$$

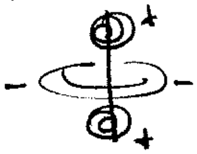
$$= 4\pi \cdot \frac{2}{15}$$

$$\int d\cos\theta \, d\phi \ (\sin^2\theta - \sin^4\theta \cos^2\phi)$$

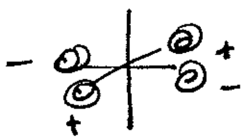
$$= 2\pi \int d\cos\theta \ (\sin^2\theta - \tfrac{1}{2}\sin^4\theta)$$

$$= 4\pi \cdot \frac{6}{15} = 4\pi \cdot \frac{2}{5}$$

so



$$P = \frac{\mu_0}{288\pi} \frac{\omega^6}{c^3} \cdot \frac{3}{10} \, |Q_{30}|^2$$



$$P = \frac{\mu_0}{288\pi} \frac{\omega^6}{c^3} \cdot \frac{2}{5} \, |Q_{10}|^2$$

both formulae correspond to

$$P = \frac{\mu_0}{288\pi} \frac{\omega^6}{c^3} \cdot \frac{1}{5} \sum_{ij} |Q_0^{ij}|^2$$

$$= \frac{\mu_0}{1440\pi} \frac{\omega^6}{c^3} \sum_{ij} |Q_0^{ij}|^2$$

which is actually the general result for any Q_0^{ij}.

Looking back at the calculations of this lecture, we found, for a localized distribution of charges moving at nonrelativistic velocities

$$\vec{E} = \frac{\mu_0}{4\pi} \, \mathrm{Re} \, \frac{e^{-i\omega t + ikx}}{x} \cdot$$

$$\cdot \, \omega^2 \cdot \left\{ (\vec{p}_0)_\perp - \frac{1}{c}(\hat{x} \times \vec{m}_0) - i\frac{k}{6}(Q_0 \cdot \hat{x})_\perp + \cdots \right\}$$

$$\vec{B} = \frac{1}{c} \, \hat{x} \times \vec{E}$$

in the far field where the E and B fields correspond to radiation. These terms are called, respectively,

- electric dipole radiation (E_1)
- magnetic dipole radiation (M_1)
- electric quadrupole radiation (E_2)

The M_1 term is the first relativistic correction, smaller by $\frac{v}{c}$.

The E_2 term is the first correction in powers of kR

The expansion can, in principle, be developed systematically to any desired order in kR.