Potentials and Fields of Point Charges

We now have a general expression for the potentials created by charges and currents:

$$A^\mu(t,\vec{x}) = \frac{\mu_0}{4\pi}\int_{-\infty}^{t} dt_y \int d^3y \; \frac{1}{|\vec{x}-\vec{y}|}\,\delta\!\left(t-t_y-\frac{|\vec{x}-\vec{y}|}{c}\right)\; j^\mu(t_y,\vec{y})$$

In the next several lectures, we will apply this equation to a variety of physical situations. To begin, let's compute the potentials due to a moving point charge.

We need to know, first, what to put for j^μ for a point charge. Let the charge follow the trajectory

$$(t,\ \vec{y}(t)) \qquad\qquad \vec{v}(t) = \frac{d\vec{y}}{dt}$$

and have charge q. Then

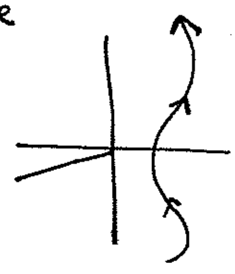
$$\rho(t,\vec{y}) = q\,\delta^{(3)}(\vec{y}-\vec{y}(t)) \qquad\qquad \vec{j} = q\,\vec{v}(t)\,\delta^{(3)}(\vec{y}-\vec{y}(t))$$

To make it obvious that this is a 4-vector, compare these expressions to

$$j^\mu(y) = \int d\sigma\ \frac{dx^\mu}{d\sigma}\ \delta^{(4)}(y-y(\sigma))$$

where $y^\mu(\sigma)$ is the trajectory in space-time

and σ is an arbitrary parameter of this trajectory



The integral is independent of which parameter we pick.

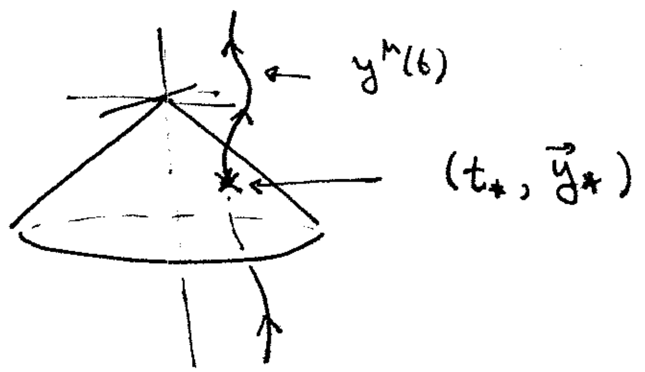
Taking $\sigma = \tau$ (proper time), j^μ is obviously a 4-vector.

Taking $\sigma = t$

$$\frac{dx^0}{dt} = c \qquad \delta^{(4)}(y - y(\sigma)) = \delta(t_y - t)\,\delta^{(3)}(\vec{y} - \vec{y}(t))$$

so we can integrate over the first δ-fnctn and recover the expression above.

Now let's compute the potentials at $(t, \vec{x})$. The current $j^\mu(t_y, \vec{y})$ is nonzero only on the particle's space-time trajectory. This trajectory intersects the backward light-cone of $(t, \vec{x})$ in precisely one point:



[The trajectory cannot intersect the cone in two points, or miss entirely, unless the particle travels at a speed greater than c.]

Call the intersection $(t_*, \vec{y}_*)$, $\vec{y}_* = \vec{y}(t_*)$, and write $\vec{V}_* = \frac{d\vec{y}}{dt}(t_*)$

Then

$$A^\mu(t,\vec{x}) = \frac{\mu_0}{4\pi}\int_{-\infty}^{t} dt_y \, \frac{1}{|\vec{x}-\vec{y}(t_y)|}\,\delta\!\left(t-t_y-\frac{|\vec{x}-\vec{y}(t_y)|}{c}\right)$$

$$\cdot\,(qc,\; q\vec{V}(t_y))$$

Finally, we must integrate over the δ-function. The δ-function is satisfied for $t_y = t_*$. Expanding the argument about this point:

$$t - t_y - \frac{|\vec{y}(t_y)-\vec{x}|}{c}$$

$$= (t-t_*) - (t_y - t_*) - \frac{|\vec{y}(t_*) + \vec{V}_*(t_y-t_*) - \vec{x}|}{c}$$

$$= (t-t_*) - (t_y-t_*) - \frac{|\vec{y}_*-\vec{x}|}{c} - \frac{\vec{V}_*(t_y-t_*)}{c}\cdot\frac{\vec{y}_*-\vec{x}}{|\vec{y}_*-\vec{x}|} + \cdots$$

now

$$t - t_* - \frac{|\vec{R}_*|}{c} = 0, \quad \text{for} \quad \vec{R}_* = \vec{x}-\vec{y}_*$$

so

$$= -(t_y-t_*)\left[1 - \frac{\vec{V}_*}{c}\cdot\frac{\vec{R}_*}{|\vec{R}_*|}\right] + O\big((t_y-t_*)^2\big)$$

then

$$\int dt_y\, \delta\!\left(t-t_y-\frac{|\vec{x}-\vec{y}(t_y)|}{c}\right) = \frac{1}{1-\frac{\vec{V}_*}{c}\cdot\frac{\vec{R}_*}{|\vec{R}_*|}}$$

and

$$A^\mu = \frac{\mu_0}{4\pi}\cdot\frac{1}{\left[R_* - \frac{\vec{V}_*}{c}\cdot\vec{R}_*\right]}\,(qc,\; q\vec{V}_*)$$

again

$$\phi(t,\vec{x}) = \frac{1}{4\pi\epsilon_0} \frac{q}{\left[R_* - \frac{\vec{V}_*}{c}\cdot\vec{R}_*\right]}$$

$$\vec{A}(t,\vec{x}) = \frac{\mu_0}{4\pi} \frac{q\vec{V}_*}{\left[R_* - \frac{\vec{V}_*}{c}\cdot\vec{R}_*\right]} \qquad \left[\frac{\mu_0}{4\pi} = \frac{1}{4\pi\epsilon_0 c^2}\right]$$

These expressions are called the Liénard - Wiechert potentials

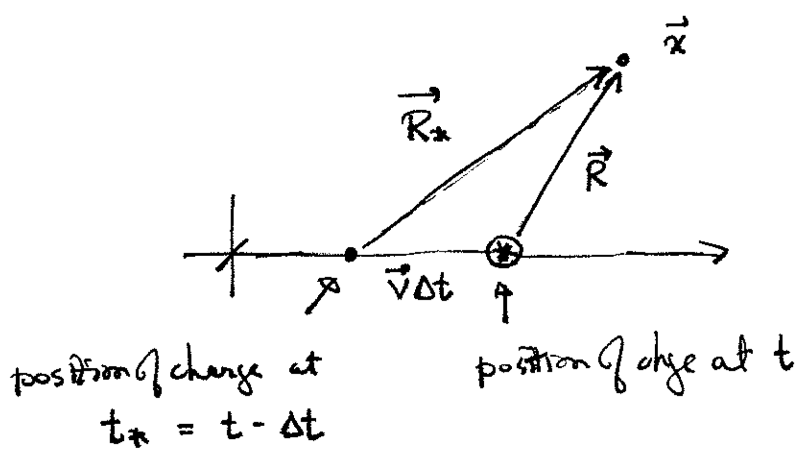
We can check these potentials by recomputing the fields of a point charge moving at constant velocity. Let the particle trajectory be

$$\vec{y}(t_y) = (0, 0, v t_y)$$

And we will compute the potentials at

$$(t, \vec{x}) = (t, x, y, z)$$

The potentials at $(t,\vec{x})$ are determined by the motion of the point charge at an earlier time



The condition for t_* is

$$|\vec{R}_*| = c\,(t-t_*) = \left[x^2+y^2+(z-vt_*)^2\right]^{1/2}$$

$$(t-t_*)^2 = \frac{1}{c^2}\left(x^2+y^2+[z-vt_*]^2\right)^{1/2}$$

$$t^2-2tt_*+t_*^2 = \frac{x^2}{c^2}+\frac{y^2}{c^2}+\frac{z^2}{c^2}-2\frac{z}{c}\frac{v}{c}t_*+\frac{v^2}{c^2}t_*^2$$

$$t_*^2(1-v^2/c^2)-2t_*\left(t-\frac{v}{c}\frac{z}{c}\right) = \frac{x^2+y^2}{c^2}+\frac{z^2}{c^2}-t^2$$

let $\gamma = \dfrac{1}{[1-v^2/c^2]^{1/2}}$ $\quad \beta = \dfrac{v}{c}$ $\quad$ note that $\gamma^2-1 = \dfrac{1}{1-\beta^2}-1$

$$= \frac{\beta^2}{1-\beta^2} = \beta^2\gamma^2$$

then

$$t_*^2-2t_*\gamma^2\left(t-\beta\frac{z}{c}\right) = \gamma^2\left\{\frac{x^2+y^2}{c^2}+\frac{z^2}{c^2}-t^2\right\}$$

complete the square:

$$\left[t_*-\gamma^2\left(t-\beta\frac{z}{c}\right)\right]^2 = \gamma^2\left\{\frac{x^2+y^2}{c^2}+\frac{z^2}{c^2}-t^2\right\}$$

$$+\gamma^4\left\{t^2-2\beta\frac{z}{c}t+\beta^2\frac{z^2}{c^2}\right\}$$

$$= \gamma^2\left\{\frac{x^2+y^2}{c^2}+\frac{z^2}{c^2}\underbrace{(1+\beta^2\gamma^2)}_{=\gamma^2}-2\gamma^2\beta\frac{z}{c}t+\underbrace{(\gamma^2-1)}_{\beta^2\gamma^2}t^2\right\}$$

$$= \gamma^2\left\{\frac{x^2+y^2}{c^2}+\gamma^2\left(\frac{z}{c}-\beta t\right)^2\right\}$$

then

$$t_* = \gamma^2 \left(t - \beta \frac{z}{c}\right) \pm \gamma \left(\frac{x^2+y^2}{c^2} + \gamma^2 \left(\frac{z}{c} - \beta t\right)^2 \right)^{\frac{1}{2}}$$

the solution with $(-)$ here is in the backward light-cone of $(t, \vec{x})$

$$|\vec{R}_*| = c\,(t - t_*)$$

$$= c\left[t\,(1-\gamma^2) + \gamma^2 \beta \frac{z}{c} + \gamma \left(\frac{x^2+y^2+\gamma^2(z-vt)^2}{c^2} \right)^{\frac{1}{2}}\right.$$

$$= \gamma^2 \beta z - \gamma^2 \beta^2 c t + c\gamma \left(\frac{x^2+y^2+\gamma^2(z-vt)^2}{c^2} \right)^{\frac{1}{2}}$$

$$= \gamma^2 \beta\,(z - vt) + \gamma\,\left(x^2+y^2 + \gamma^2(z-vt)^2 \right)^{\frac{1}{2}}$$

$$\frac{\vec{V}_*}{c} \cdot \vec{R}_* = \beta\,(z - vt_*)$$

$$= \beta z - \beta^2 c t_*$$

$$= \beta z - \beta^2 \gamma^2 c t + \beta^3 \gamma^2 z + \beta^2 \gamma \left(x^2+y^2+\gamma^2 (z-vt)^2\right)^{\frac{1}{2}}$$

so

$$\left(R_* - \frac{\vec{V}_*}{c} \cdot \vec{R}_*\right) = \underbrace{(\gamma^2 - 1)\beta z - \beta^3 \gamma^2 z}_{0} - \gamma^2 \beta^2 \underbrace{ct + \beta \gamma^2}_{=0} ct$$

$$+ \gamma(1-\beta^2)\left[x^2+y^2+\gamma^2(z-vt)^2\right]^{\frac{1}{2}}$$

$$= \frac{1}{\gamma}\left[x^2+y^2+\gamma^2(z-vt)^2\right]^{\frac{1}{2}}$$

so, finally,

$$\phi = \frac{1}{4\pi\epsilon_0} \frac{q\gamma}{[x^2+y^2+\gamma^2(z-vt)^2]^{1/2}}$$

$$\vec{A} = \frac{1}{4\pi\epsilon_0 c^2} \frac{q\gamma\,(0,0,v)}{[x^2+y^2+\gamma^2(z-vt)^2]^{1/2}}$$

then

$$-\vec{\nabla}\phi = \frac{1}{4\pi\epsilon_0} \frac{q\gamma}{[x^2+y^2+\gamma^2(z-vt)^2]^{3/2}} (x, y, \gamma^2(z-vt))$$

$$-\frac{\partial\vec{A}}{\partial t} = \frac{1}{4\pi\epsilon_0} \frac{q\gamma\,(0,0,v/c^2)}{[x^2+y^2+\gamma^2(z-vt)^2]^{3/2}} (-\gamma^2 v\,(z-vt))$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{4\pi\epsilon_0 c^2} \frac{q\gamma}{[x^2+y^2+\gamma^2(z-vt)^2]^{3/2}} (-y, x, 0)$$

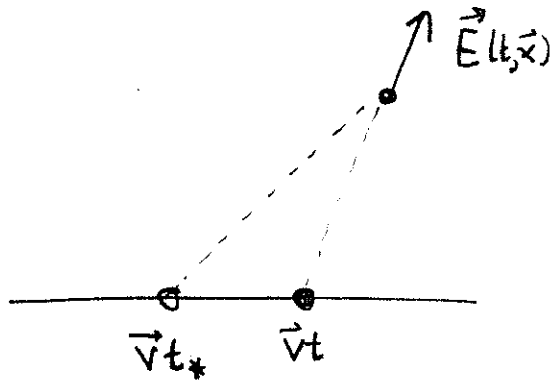
so

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q\gamma\,(x, y, (z-vt))}{[x^2+y^2+\gamma^2(z-vt)^2]^{3/2}}$$

$$\vec{B} = \frac{\vec{v}}{c^2}\times\vec{E}$$

Just as we found last week by boosting the Coulomb field.

As we have already discussed, although the electric fields point back to the instantaneous location of the charge, the field is actually set up by the motion of the charge well upstream:



If the charge *changes* its velocity, there is a disruption in the $\vec{E}$ and $\vec{B}$ fields which results in the production of *transverse* $\vec{E}$ and $\vec{B}$ fields — radiation. We can compute these fields formally by differentiating the Liénard-Wiechert potentials for the case of a more general velocity.

To do this, we need to compute derivatives of $\vec{R}_*$ and $\vec{V}_*$. This is slightly complicated, because these quantities depend on t_*, defined as the solution to

$$t - t_* - \frac{|\vec{x} - \vec{y}(t_*)|}{c} = 0$$

Differentiating this equation gives

$$dt - dt_* - \left(\frac{d\vec{x} - \vec{V}\,dt_*}{c}\right) \cdot \frac{(\vec{x} - \vec{y}_*)}{|\vec{x} - \vec{y}_*|} = 0$$

so that

$$-\frac{\partial t}{\partial x^i} - \frac{1}{c}\frac{R_*^i}{R_*} + \frac{\partial t}{\partial x^i}\vec{\beta}_* \cdot \frac{\vec{R}_*}{R_*} = 0$$

and
$$1 - \frac{\partial t_*}{\partial t} + \frac{\partial t_*}{\partial t}\vec{\beta}_* \cdot \frac{\vec{R}_*}{R_*} = 0$$

where I write $\vec{\beta}_* = \frac{\vec{V}_*}{c}$ $\quad R_* = |\vec{R}_*|$

These equations imply

$$\vec{\nabla} t_* = -\frac{1}{c}\frac{\vec{R}_*}{(R_* - \vec{\beta}_* \cdot \vec{R}_*)}$$

$$\frac{\partial t_*}{\partial t} = \frac{R_*}{R_* - \vec{\beta}_* \cdot \vec{R}_*}$$

From these relations, we can compute the derivatives of $\vec{R}_*$ and $\vec{V}_*$. Indices will proliferate, so I will now switch to index notation

$$\nabla^i R_* = \nabla^i c(t - t_*) = \frac{R_*^i}{R_* - \vec{\beta}_* \cdot \vec{R}_*}$$

$$\frac{\partial R_*}{\partial t} = \frac{\partial}{\partial t} c(t - t_*) = c\left(1 - \frac{R_*}{R_* - \vec{\beta}_* \cdot \vec{R}_*}\right)$$

$$= -c\,\frac{\vec{\beta}_* \cdot \vec{R}_*}{R - \vec{\beta}_* \cdot \vec{R}_*}$$

$$\nabla^i R_*^j = \frac{\partial}{\partial x^i}\left(x^j - y^j(t_*)\right)$$

$$= \delta^{ij} - \frac{dy^j}{dt_*}\frac{\partial t_*}{\partial x^i}$$

$$= \delta^{ij} - v_*^j \left(-\frac{1}{c}\right) \frac{R_*^i}{R_* - \vec{\beta}_* \cdot \vec{R}_*}$$

or

$$\nabla^i R_*^j = \frac{\delta^{ij}(R_* - \vec{\beta}_* \cdot \vec{R}_*) + \beta_*^j R_*^i}{R_* - \vec{\beta}_* \cdot \vec{R}_*}$$

$$\frac{\partial R_*^j}{\partial t} = \frac{\partial}{\partial t}\left(x^j - y^j(t_*)\right)$$

$$= - v_*^j \frac{\partial t_*}{\partial t}$$

$$\frac{\partial R_*^j}{\partial t} = - v_*^j \frac{R_*}{R_* - \vec{\beta}_* \cdot \vec{R}_*}$$

$$\frac{\partial v_*^j}{\partial t} = \frac{d^2 y^j}{dt_*^2}\frac{\partial t_*}{\partial t}$$

$$= a_*^j \frac{R_*}{R_* - \vec{\beta}_* \cdot \vec{R}_*}$$

where $\vec{a}_*$ is the instantaneous acceleration of the particle at t_*

$$\nabla^i v_*^j = \frac{d^2 y^j}{dt_*^2}\frac{\partial t_*}{\partial x^i} = -\frac{1}{c}\, a_*^j \frac{R_*^i}{R_* - \vec{\beta}_* \cdot \vec{R}_*}$$

It will be useful also to compute

$$\nabla^i (R_* - \vec{\beta}_* \cdot \vec{R}_*)$$

$$= \frac{1}{(R_* - \vec{\beta}_* \cdot \vec{R}_*)} \Big\{ R_*^i - \beta_*^j \left(\delta^{ij} (R_* - \beta_*^k R_*^k) + \beta_*^j R_*^i \right) + \frac{1}{c^2} a_*^j R_*^i \cdot R_*^j \Big\}$$

$$= \frac{1}{(R_* - \vec{\beta}_* \cdot \vec{R}_*)} \Big\{ R_*^i (1 - \beta_*^2) - \beta_*^i (R_* - \beta_*^k R_*^k) + \frac{1}{c^2} R_*^i \, a_*^j R_*^j \Big\}$$

$$\frac{1}{c} \frac{\partial}{\partial t} (R_* - \vec{\beta}_* \cdot \vec{R}_*)$$

$$= \frac{1}{(R_* - \vec{\beta}_* \cdot \vec{R}_*)} \Big\{ -\vec{\beta}_* \cdot \vec{R}_* - \beta_*^k (-\beta_*^k R_*) - \left(\frac{a_*^k R_*}{c^2} \right) R_*^k \Big\}$$

$$= \frac{1}{(R_* - \vec{\beta}_* \cdot \vec{R}_*)} \Big\{ \beta_*^k R_*^k + \beta_*^2 R_* - \frac{1}{c^2} a_*^k R_*^k R_* \Big\}$$

then:

$$-\nabla^i \phi = -\nabla^i\left[\frac{q}{4\pi\epsilon_0}\,\frac{1}{(R_* - \vec{\beta}_*\cdot\vec{R}_*)}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\,\frac{1}{(R_* - \vec{\beta}_*\cdot\vec{R}_*)^3}\Big\{R_*^i(1-\beta_*^2) - \beta_*^i(R_* - \beta_*^j R_*^j) + \frac{1}{c^2}a_*^j R_*^j R_*^i\Big\}$$

$$-\frac{\partial}{\partial t}A^i = -\frac{\partial}{\partial t}\left[\frac{q}{4\pi\epsilon_0 c^2}\,\frac{V_*^i}{(R_* - \vec{\beta}_*\cdot\vec{R}_*)}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\,\frac{1}{(R_* - \vec{\beta}_*\cdot\vec{R}_*)^3}\Big\{\beta_*^i\big(\beta_*^2 R_* \text{ [redacted] } \beta_*^j R_*^j - \frac{1}{c^2}a_*^j R_*^j R_*\big) - \frac{a_*^i}{c^2}R_*(R_* - \beta_*^j R_*^j)\Big\}$$

so!

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\,\frac{1}{(R_* - \vec{\beta}_*\cdot\vec{R}_*)^3}$$

$$\cdot\Big\{R_*^i(1-\beta_*^2) - \beta_*^i R_*(1-\beta_*^2) + \cancel{\beta_*^i\beta_*^j R_*^j} - \cancel{\beta_*^i\beta_*^j R_*^j} + \frac{1}{c^2}\big(a_*^j R_*^j R_*^i - a_*^i R_*^2 + a_*^i R_*\beta_*^j R_*^j - a_*^j R_*^j R_*\beta_*^i\big)\Big\}$$

Let $\vec{\mathcal{R}} = \vec{R}_* - \vec{\beta}_* R_*$

then

$$\vec{R}_* \times (\vec{\mathcal{R}} \times \vec{a}_*)$$

$$= \vec{R}_* \times (\vec{R}_* \times \vec{a}_*) - \vec{R}_* \times (\vec{\beta}_* \times \vec{a}_*) R_*$$

$$= \vec{R}_* (\vec{R}_* \cdot \vec{a}_*) - \vec{a}_* R_*^2 - \vec{\beta}_* R_* \vec{R}_* \cdot \vec{a}_*$$

$$+ \vec{a}_* R_* (\vec{\beta}_* \cdot \vec{R}_*)$$

and this is exactly the structure of the a_* term in the expression for $\vec{E}$. Thus:

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1}{(R_* - \vec{\beta}_* \cdot \vec{R}_*)^3} \left[(1-\beta_*^2)\vec{\mathcal{R}} + \frac{1}{c^2}\vec{R}_* \times (\vec{\mathcal{R}} \times \vec{a}_*) \right]$$

To compute $\vec{B}$ we need

$$\epsilon^{ijk} \nabla^j A^k = \epsilon^{ijk} \nabla^j \left[\frac{q}{4\pi\epsilon_0 c^2} \frac{V_*^k}{(R_* - \vec{\beta}_* \cdot \vec{R}_*)} \right]$$

$$= \epsilon^{ijk} \frac{q}{4\pi\epsilon_0 c} \frac{1}{(R_* - \vec{\beta}_* \cdot \vec{R}_*)^3}$$

$$\cdot \Big\{ \big(-R_*^j (1-\beta_*^2) + \beta_*^j (R_* - \beta_*^l R_*^l) - \tfrac{1}{c^2} R_*^j a_*^l R_*^l\big) \beta_*^k$$

$$- \tfrac{1}{c^2} a_*^k R_*^j (R_* - \beta_*^l R_*^l) \Big\}$$

$$= \frac{q}{4\pi\epsilon_0 c} \frac{1}{(R_* - \vec{\beta}_* \cdot \vec{R}_*)^3}$$

$$\Big\{ \vec{\beta}_* \times \vec{R}_* (1-\beta_*^2)$$

$$+ \frac{1}{c^2}\big(- \vec{R}_* \times \vec{\beta}_* (\vec{R}_* \cdot \vec{a}_*) + \vec{R}_* \times \vec{a}_* (\vec{\beta}_* \cdot \vec{R}_* - R_*)\big) \Big\}$$

The first line in the bracket is

$$\vec{R}_* \times (-\vec{\beta}_*)(1-\beta_*^2) = \hat{R}_* \times \vec{\mathcal{R}}(1-\beta_*^2)$$

comparing the second line to the top of p. 13 we see ~~that~~ this is

$$\frac{1}{c^2} \hat{R}_* \times [\vec{R}_* \times (\vec{\mathcal{R}} \times \vec{a}_*)]$$

in all

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1}{(R_* - \vec{\beta}_* \cdot \vec{R}_*)^3} \Big\{ (1-\beta_*^2)\vec{\mathcal{R}} + \frac{1}{c^2} \vec{R}_* \times (\vec{\mathcal{R}} \times \vec{a}_*) \Big\}$$

$$\vec{B} = \frac{1}{c} \hat{R}_* \times \vec{E}$$

with $$\vec{\mathcal{R}} = \vec{R}_* - \vec{\beta}_* R_*$$

Note that $\vec{B}$ is perpendicular to $\vec{E}$ and perpendicular to $\vec{R}_*$.

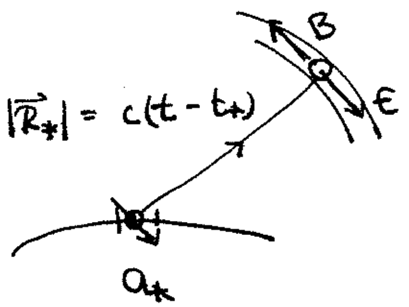
These formulae have a very interesting structure. They have, first, a term independent of acceleration. We have already seen ~~that~~ this term sets up the appropriately boosted Coulomb field. As $R_* \to \infty$, this term of E and B goes to zero as

$$\text{indep of } \vec{a}_*: \qquad E, B \sim \frac{1}{R_*^2} .$$

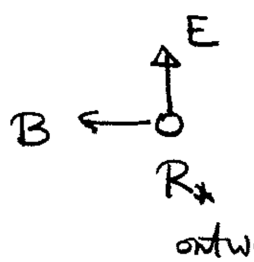
The second term is proportional to the acceleration. In this term, the $\vec{E}$ field is perpendicular to $\vec{R}_*$, and $\vec{B}$ is perpendicular to both $\vec{R}_*$ and $\vec{E}$, and:

$$E, B \sim \frac{1}{R_*}$$

so:



Seen head-on, the outward moving $\vec{E}$ and $\vec{B}$ field configuration looks like



with $|\vec{E}| = c|\vec{B}|$

so this is exactly an outward moving electromagnetic wave. If $E, B \sim \frac{1}{R_*}$, then

$$S = \frac{1}{\mu_0} \vec{E} \times \vec{B} \sim \frac{1}{R_*^2}$$

so that $\int d^2x \; \hat{n} \cdot \vec{S}$ is constant as the sphere of radiation moves outward; the pulse of electromagnetic radiation carries a finite amount of energy.

Let's carry out the computation of the fields and energy in radiation more explicitly for some special cases. Consider first the simplest case where the particle emitting the radiation is instantaneously <u>at rest</u> at the time that it is hit by a force and experiences an acceleration.

Then $\vec{\beta}_* = 0$, so that

$$(R_* - \vec{\beta}_* \cdot \vec{R}_*) = R_*$$

$$\vec{\mathcal{R}} = \vec{R}_* - \vec{\beta}_* R_* = \vec{R}_*$$

so at a distance $\vec{R}_*$ or $\vec{R}$

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1}{R^3} \frac{\vec{R} \times (\vec{R} \times \vec{a})}{c^2}$$

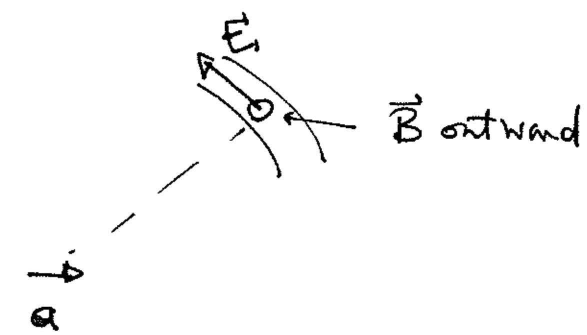
$$\vec{B} = \frac{1}{c} \hat{R} \times \vec{E}$$

Then

$$\vec{E} = \frac{q}{4\pi\epsilon_0 c^2} \frac{1}{R^3}\left[\vec{R}\,(\vec{R}\cdot\vec{a}) - \vec{a}\,R^2\right]$$

$$= -\frac{q\mu_0}{4\pi}\frac{1}{R}\left[\vec{a} - \hat{R}(\hat{R}\cdot\vec{a})\right]$$

so $\vec{E}$ is parallel to the component of $\vec{a}$ perpendicular to $\vec{R}$.



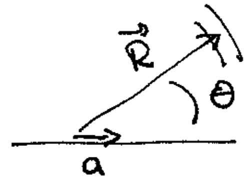
The energy flow outward is given by

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$$= \frac{1}{\mu_0}\left(\frac{q\mu_0}{4\pi}\right)^2 \frac{1}{R^2 c}\left(\vec{a} - \hat{R}\;\hat{R}\cdot\vec{a}\right)^2 \;\hat{R}$$

If the acceleration defines the $\hat{z}$ axis

$$\vec{a} = a\,\hat{z}$$



then

$$\left(\vec{a} - \hat{R}\;\hat{R}\cdot\vec{a}\right)^2 = a^2 - 2(\hat{R}\cdot\vec{a})^2 + (\hat{R}\cdot\vec{a})^2$$

$$= a^2 - (\hat{R}\cdot\vec{a})^2$$

gives $$(\vec{a} - \hat{R}\,\hat{R}\cdot\vec{a})^2 = a^2 - a^2\cos^2\theta$$
$$= a^2 \sin^2\theta$$

$$\vec{S} = \frac{q^2 \mu_0}{16\pi^2 c} \frac{a^2 \sin^2\theta}{R^2} \hat{R}$$

then the energy flowing through a spherical surface per unit time is

$$\text{Power} = J/\text{sec} = \int d^2x \; \hat{n}\cdot\vec{S}$$
$$= \int d\cos\theta \, d\phi \; R^2 \cdot \frac{q^2 \mu_0}{16\pi^2 c} \frac{a^2 \sin^2\theta}{R^2}$$

$$= \int d\cos\theta \, d\phi \; \frac{\mu_0 q^2}{16\pi^2 c} a^2 \sin^2\theta$$

We can write this as

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c} a^2 \sin^2\theta$$

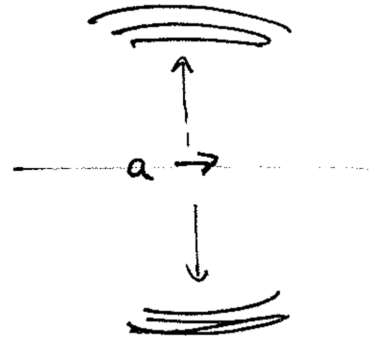
or ~~integrate~~ over solid angle

$$\int d\cos\theta \, d\phi \; \sin^2\theta = \int_{-1}^{1} d\cos\theta \, (1-\cos^2\theta) \int_0^{2\pi} d\phi$$
$$= (2 - \tfrac{2}{3})\, 2\pi = \frac{8\pi}{3}$$

so that

$$\mathcal{P} = \frac{\mu_0 q^2}{6\pi c} |\vec{a}|^2$$

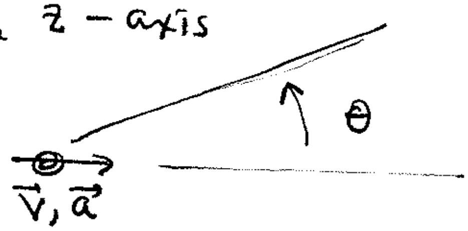
This is Larmor's formula. It is important to note that most of the radiation is in directions orthogonal to the direction of $\vec{a}$:



We will study the generalities of radiation patterns in more detail next week.

It is ~~interest~~ to compute the radiation formulae also for the case in which the particle is moving at the time of the acceleration, with $\vec{a} \parallel \vec{v}$. This is another very straightforward application of the formulae on p. 14.

Let $\vec{a} \parallel \vec{v}$ define the z-axis



$$(R_* - \vec{\beta}_* \cdot \vec{R}_*) = R_*(1 - \beta_* \cos\Theta)$$

$$\vec{\beta}_* \times \vec{a}_* = 0 \text{, so} \quad \vec{R}_* \times (\vec{R} \times \vec{a}_*) = \vec{R}_* \times (\vec{R}_* \times \vec{a})$$

so

$$\vec{E} = \frac{\mu_0 q}{4\pi} \frac{1}{R^2} \frac{1}{(1-\beta_* \cos\Theta)^3} \vec{R} \times (\vec{R} \times \vec{a})$$

comparing to the previous case, again

$\vec{E}$ is parallel to the component of $\vec{a}$ perpendicular to $\vec{R}$

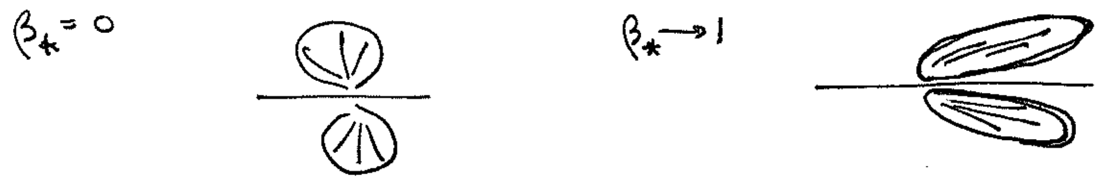
$$|\vec{E}| = \frac{\mu_0 q}{4\pi} \frac{1}{(1-\beta_* \cos\Theta)^3} |\vec{a}| \sin\Theta$$

$$|\vec{B}| = \frac{1}{c} |\vec{E}|$$

$$\vec{S} = \frac{\mu_0 q^2}{16\pi^2 c} \frac{1}{(1-\beta_* \cos\Theta)^6} a^2 \sin^2\Theta \frac{\hat{R}}{R^2}$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c} \frac{a^2 \sin^2\Theta}{(1-\beta_* \cos\Theta)^6}$$

The radiation pattern has an exact zero in the forward direction, but for high-velocity ($\beta_* \to 1$), the denominator generates a distribution that is more and more forward-peaked



For $\beta_* \to 1$

$$\frac{\sin^2\theta}{(1-\beta_*\cos\theta)^6} \sim \frac{\theta^2}{\left[1-\beta_* + \frac{1}{2}\theta^2\right]^6}$$

now $\quad \gamma_*^{-2} = 1-\beta_*^2 \cong 2(1-\beta_*) \quad$ so $\quad 1-\beta_* \approx \frac{1}{2}\gamma_*^{-2}$

$$\sim \frac{64\,\gamma_*^{12}\,\theta^2}{\left[1+\gamma_*^2\theta^2\right]^6}$$

From this formula, it is clear that the forward peak of the radiation results from a forward boost of the $\beta = 0$ radiation pattern.

The formula for $dP/d\Omega$ above refers to the power crossing a sphere about $R=0$ at time t. To compute the energy <u>lost</u> by the particle per unit time we must compute

$$\frac{dP_{lost}}{d\Omega} = \frac{dP}{d\Omega} \cdot \frac{dt}{dt_*}$$

now $$\frac{dt_*}{dt} = \frac{R_*}{R_* - \vec{\beta}_* \vec{R}_*} = \frac{1}{(1-\beta_* \cos\theta)}$$

so

$$\frac{dP_{lost}}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c} \frac{a^2 \sin^2\theta}{(1-\beta_* \cos\theta)^5}$$

to be integrated over solid angle.

$$P_{lost} = \frac{\mu_0 q^2}{6\pi c} |\vec{a}|^2 \frac{1}{(1-\beta^2)^3}$$