Solution of the Relativistic Wave Equation

By introducing the potentials $A^\mu = (\Phi/c, \vec{A})$, we reduced Maxwell's equations, in Lorentz gauge, to

$$\Box A^\mu = \mu_0 J^\mu$$

This reduces the problem of computing radiation — the electromagnetic fields which respond to a current — to the solution of the scalar wave equation

$$\Box \phi(x) = \sigma(x)$$

In this lecture, I would like to solve this equation. The best technique is to use a Green's function method: Solve the equation

$$\Box G(x;y) = \delta^{(4)}(x-y)$$

and then write the solution of the more general equation as

$$\phi(x) = \int d^4y \; G(x;y)\,\sigma(y)$$

Indeed

$$\Box \phi(x) = \int d^4y \; \Box G(x;y)\,\sigma(y)$$

$$= \int d^4y \; \delta^{(4)}(x-y)\,\sigma(y)$$

$$= \sigma(x) \quad \checkmark$$

Griffiths solves the Green's function equation by guessing the answer. I would like to use a constructive method that uses some of the results on Fourier analysis that we discussed in 121.

As a warm-up, let's compute the Green's function for the Laplace operator

$$-\nabla^2 G(x;y) = \delta^{(3)}(x-y)$$

or for the generalization

$$[-\nabla^2 + \mu^2]\, G(x;y) = \delta^{(3)}(x-y)$$

which represents the static behavior of the Klein-Gordon equation

$$[\Box + \mu^2]\, \phi(x) = \sigma(x)$$

The problem is translation invariant, so $G(x;y) = G(x-y)$, and now we can set $y=0$. Introduce a Fourier representation for $G(x)$:

$$G(x) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, \tilde{G}(k)$$

then

$$\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}} \left\{ [-(i\vec{k})^2 + \mu^2]\, \tilde{G}(k) = 1 \right\}$$

or

$$\tilde{G}(k) = +\frac{1}{k^2+\mu^2}$$

To recover $G(x)$, we just have to do some integrals:

$$G(x) = \int \frac{d^3k}{(2\pi)^3} \, e^{i\vec{k}\cdot\vec{x}} \, \frac{1}{k^2+\mu^2}$$

Choose $\vec{x} = (0,0,x)$ and represent the integral in spherical coordinates:

$$G(x) = \int \frac{dk \, k^2 \, d\cos\theta \, d\phi}{8\pi^3} \, e^{ikx\cos\theta} \, \frac{1}{k^2+\mu^2}$$

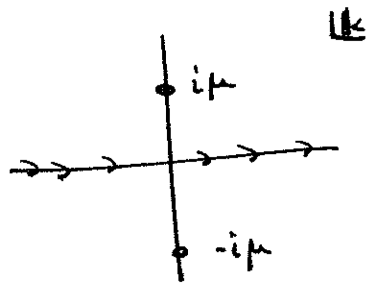
$$= \frac{2\pi}{8\pi^3} \int_0^\infty dk \, k^2 \, \frac{1}{k^2+\mu^2} \int_{-1}^{1} d\cos\theta \, e^{ikx\cos\theta}$$

$$= \frac{1}{4\pi^2} \int_0^\infty dk \, k^2 \frac{1}{k^2+\mu^2} \, \frac{1}{ikx} \left(e^{ikx} - e^{-ikx}\right)$$

Now there is a neat trick: the two terms combine into an integral dk from $+\infty$ to $-\infty$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk \, k \, \frac{1}{k^2+\mu^2} \, e^{ikx} \cdot \frac{1}{ix}$$

Now this is an ~~integral~~ we can do by contour methods:



For $x > 0$ we can close the contour in the upper half plane:

$$= \frac{1}{4\pi^2} \; 2\pi i \; i\mu \; \frac{1}{2i\mu} \; e^{-\mu x} \; \frac{1}{ix}$$

so

$$G(\vec{x}) = \frac{1}{4\pi} \frac{1}{|\vec{x}|} e^{-\mu|\vec{x}|}$$

For $\mu=0$, we obtain the familiar Coulomb potential. For $\mu>0$, we obtain the "Yukawa potential". This is the potential due to a field with massive quanta, where μ is related to the mass m through

$$\mu = \frac{mc}{\hbar}$$

Now return to

$$\Box G(x) = \delta^{(4)}(x) .$$

Introduce a Fourier representation

$$G(x) = \int \frac{d\omega \, d^3k}{(2\pi)^4} e^{-i\omega t} e^{i\vec{k}\cdot\vec{x}} \tilde{G}(\omega, \vec{k})$$

then

$$\left[-\frac{\omega^2}{c^2} + (\vec{k})^2\right] \tilde{G}(\omega, k) = 1$$

so

$$G(t,\vec{x}) = \int \frac{d\omega}{2\pi}\frac{d^3k}{(2\pi)^3}\, e^{-i\omega t}\, e^{i\vec{k}\cdot\vec{x}}\, \frac{1}{[-\omega^2/c^2 + k^2]}$$

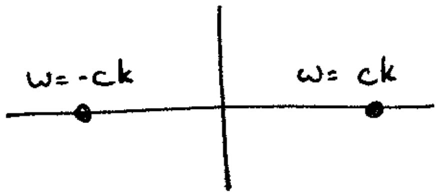
we can use the rotation symmetry of the problem to set

$$\vec{x} = (0, 0, x) \qquad x > 0$$

Let's do the ω integral first. There are two poles, at

$$\omega = \pm ck,$$

on the real axis.

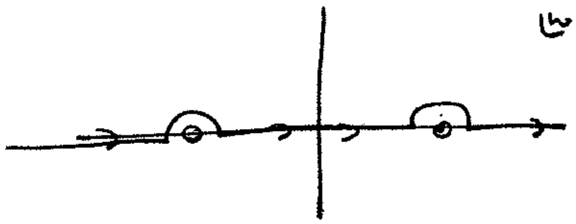


We can choose the ω contour to go above or below these poles (or choose some compromise between these positions) As in the oscillator problems that we studied last term, the correct choice is given by the physical requirement that the response $G(x)$ occurs only after the source appears at $t=0$ $x=0$.

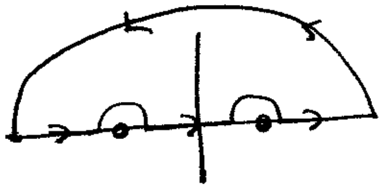
That is, we want the retarded Green's function

$$G(t,x) = \begin{cases} 0 & t < 0 \\ (\text{something}) & t > 0 \end{cases}$$

The corresponding choice of contour is

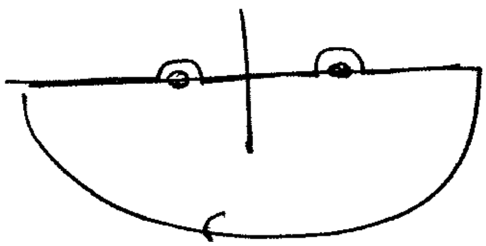


for $t<0$ we close upward:



$= 0$

for $t>0$, we close the contour downward



$=$

$$\int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{1}{-\omega^2/c^2 + k^2}$$

$$= \left(-\frac{2\pi i}{2\pi}\right)\left(-\frac{1}{c^2}\right)^{-1} \left\{ \frac{e^{-ikct}}{2kc} + \frac{e^{ikct}}{-2kc} \right\}$$

$$= \frac{i}{2k} c^2 \left[e^{-ikct} - e^{ikct} \right]$$

As before, the ~~integral~~ d^3k can be converted to polar coordinates

$$\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} = \int \frac{dk\, k^2\, d\cos\theta\, d\phi}{(2\pi)^3} e^{ikx\cos\theta}$$

$$= \frac{1}{4\pi^2} \int_0^\infty dk\, k^2 \frac{1}{ikx} \left(e^{ikx} - e^{-ikx}\right)$$

so, in all, we find

$$G(t,x) = \frac{c}{8\pi^2}\frac{1}{x}\int_0^\infty dk\,(e^{-ikct} - e^{ikct})(e^{ikx} - e^{-ikx})$$

$t>0$

$$= \frac{c}{8\pi^2}\frac{1}{x}\int_{-\infty}^{\infty} dk\,(e^{-ik(ct-x)} - e^{-ik(ct+x)})$$

$$= \frac{1}{8\pi^2}\frac{1}{x}\,c\,2\pi\,[\delta(ct-x) - \delta(ct+x)]$$

Since $t>0$, $x>0$, we can drop the second δ-function. Also

$$c\,\delta(ct-x) = \delta(t-\tfrac{x}{c})$$

so, finally

$$G(t,|\vec{x}|) = \begin{cases} 0 & t<0 \\ \dfrac{1}{4\pi}\dfrac{1}{|\vec{x}|}\,\delta\!\left(t-\dfrac{|\vec{x}|}{c}\right) & t>0 \end{cases}$$

and the solution to the more general scalar wave problem is

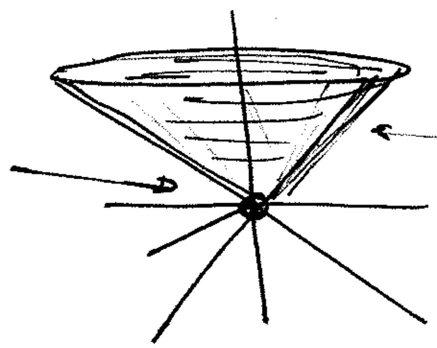
$$\phi(t,\vec{x}) = \int_{-\infty}^{t} dt_y \int d^3y\;\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{y}|}\,\delta\!\left(t-t_y-\frac{|\vec{x}-\vec{y}|}{c}\right)$$

$$\cdot\;\sigma(t_y,\vec{y})$$

The ~~integration~~ range for t_y represents the retarded boundary condition:

$$\text{no response unless } t > t_y.$$

Actually, the result we found has the amazing property that the only response is on the forward light-cone:



In principle, there would be no problem if there were a response for $t > \frac{|\vec{x}|}{c}$. But if $G(x)$ were nonzero for $0 < t < \frac{|\vec{x}|}{c}$, we would have a serious problem. In that case, there would be a frame in which the response occurred for $t' < 0$, since outside the light-cone the temporal order of events depends on the frame. So it is fortunate that the wave equation itself ~~solves~~ this problem and guarantees that there is no response not only for $t < 0$ but also for $t < \frac{|\vec{x}|}{c}$!

We can apply this result to the solution of Maxwell's equations of the form

$$\Box A^\mu = \mu_0 J^\mu$$

Using again the retarded Green's function of the wave operator, we have

$$A^\mu(t,\vec{x}) = \int_{-\infty}^{t} dt_y \int d^3y \, \frac{1}{4\pi|\vec{x}-\vec{y}|} \, \delta(t-t_y-|\vec{x}-\vec{y}|)$$

$$\cdot \, \mu_0 J^\mu(t_y,\vec{y})$$

In deriving the above equation, we assumed that A^μ was in the Lorentz gauge, so we must check that the solution is consistent with this. Indeed:

$$\begin{aligned}
\partial_\mu A^\mu &= \int d^4y \left[\frac{\partial}{\partial x^\mu} G(x-y)\right] J^\mu(y) \\
&= \int d^4y \left[-\frac{\partial}{\partial y^\mu} G(x-y)\right] J^\mu(y) \\
&= \int d^4y \; G(x-y) \, \frac{\partial}{\partial y^\mu} J^\mu(y) \\
&= 0
\end{aligned}$$

because the current $J^\mu(y)$ is conserved.

Apply this result to static situations:

for electrostatics:

$$J^\mu(t_y, \vec{y}) = (\rho(\vec{y})c, \vec{0})$$

independent of time

then

$$A^\mu = \int dt_y \int d^3y \, \frac{1}{4\pi|\vec{x}-\vec{y}|} \, \mu_0 c \left(\rho(\vec{y}), \vec{0}\right) \delta\!\left(t - t_y - \frac{|\vec{x}-\vec{y}|}{c}\right)$$

so $\vec{A} = 0$, $\phi = cA^0$ is:

$$\phi(\vec{x}) = \mu_0 c^2 \int d^3y \, \frac{1}{4\pi|\vec{x}-\vec{y}|} \rho(\vec{y})$$

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3y \, \frac{1}{|\vec{x}-\vec{y}|} \rho(\vec{y})$$

similarly, for magnetostatics

$$J^\mu(t_y, \vec{y}) = (0, \vec{J}(y)) \quad \text{independent of } t_y$$

now $\phi = 0$,

$$\vec{A}(x) = \frac{\mu_0}{4\pi} \int d^3y \, \frac{1}{|\vec{x}-\vec{y}|} \vec{J}(y)$$

with both results in agreement with Physics 120.