Derivation of Maxwell's Equations

Now that we have assembled all of the ingredients of relativistic electrodynamics, I would like to explain how to derive Maxwell's equations from this perspective. My derivation will involve some new postulates that we have not seen before. However, these turn out to be the postulates that lead most naturally to the fundamental laws for the other interactions of physics. So, as you study more advanced topics in physics, you will see these arguments repeated and generalized.

The first postulate is:

(1) Laws of Nature should follow from a variational principle

$$\delta S = 0$$

You have seen the particle version of this law in mechanics:

Let $S = \int dt\, L(x, \dot{x})$, with $L = \frac{1}{2}m\dot{x}^2 - V(x)$,

then the condition that S is stationary with respect to an arbitrary variation of $x(t)$ implies Newton's law

$$m\ddot{\vec{x}} = -\vec{\nabla} V$$

If we wish to derive field equations, we should write

$$S = \int d^4x \; \mathcal{L}(\phi, \partial_\mu \phi)$$

where $d^4x = c\,dt\; d^3\vec{x}$, $\phi(x)$ is the field, and $\partial_\mu \phi$ is a general derivative with respect to $\vec{x}^i$ or t. The function $\mathcal{L}$ is called the "Lagrange density". The simple relativistic field equation

$$\Box \phi = 0$$

follows from the variational principle

$$\delta S = 0 \quad \text{with} \quad S = \int d^4x \; \tfrac{1}{2}(\partial_\mu \phi \, \partial^\mu \phi)$$

Explicitly, vary S by an arbitrary variation of ϕ:

$$0 = \delta S = \int d^4x \quad \partial_\mu \phi \;\; \partial^\mu (\delta\phi)$$

Now integrate by parts. For a well-posed problem, the surface term at ∞ should vanish

$$0 = \int d^4x \;\; \delta\phi(x) \left[- \partial_\mu \partial^\mu \phi \right]$$

This is indeed zero for an arbitrary $\delta\phi$ if

$$- \partial_\mu \partial^\mu \phi = 0.$$

The derivation of field equations from a variational principle has two important advantages:

First, it is easier to insure that the equations which result are covariant. If we work from the field equations themselves, we must check carefully that these equations are covariant, that is, that the boost of a solution is also a solution. However, this result follows from the condition that the Lagrange density is <u>invariant</u> to boosts (or any other symmetry). The logic is simple: If the Lagrange density is invariant, then if $\phi(x)$ gives a stationary point of $S[\phi]$, the symmetry transform $\phi'(x)$ must equally well give a stationary point.

<u>Second</u>, it is easier to insure energy-momentum conservation. When we studied energy and momentum conservation in Maxwell's equations in 121, we needed to check carefully that the energy lost from the electromagnetic fields was gained by the particles to which they couple

$$\left[\text{eg.} \quad \frac{d}{dt} \int d^3x \, \mathcal{E} = \int d^3x \left(-\vec{J} \cdot \vec{E} \right) \right]$$

If both the particles and the electromagnetic fields are described by a variational principle, and if this expression has no preferred origin of time and space, then automatically the total energy and momentum of the system is conserved. This result — which I will not prove — is called <u>Noether's theorem</u>. Noether's theorem also gives a <u>construction</u> of the energy and momentum — because of their relation to time and space, these quantities naturally form a 4-vector.

The second postulate is

(2) The Schrödinger equation has a local gauge invariance

The postulate of gauge invariance will be different here from the one in the previous lecture. The Schrödinger equation, the basic equation of quantum mechanics, involves a complex-valued field $\psi(x)$:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi$$

The square of ψ, $|\psi(\vec{x})|^2$, gives the probability density to find the quantum particle at $\vec{x}$. The phase of ψ does not have an obvious interpretation. In fact, the Schrödinger equation is invariant to a global phase rotation

$$\psi(x) \to e^{i\alpha} \psi(x)$$

I would like to postulate that the Schrödinger equation should be replaced by an equation with local gauge invariance, i.e. invariance to the local phase rotation

$$\psi(x) \to e^{i\alpha(x)} \psi(x)$$

where $\alpha(x)$ is an arbitrary function.

[The Schrödinger equation is valid only in nonrelativistic dynamics. However, in relativistic quantum mechanics, we can replace the Schrödinger equation with a relativistic field equation with a complex-valued field, and the same considerations will apply.]

The Lagrangian description of the Schrödinger equation is

$$S = \int d^4x \left\{ \psi^* \left(-i\hbar \frac{\partial}{\partial t}\right) \psi + \frac{1}{2} \frac{\hbar^2}{m} \vec{\nabla}\psi^* \cdot \vec{\nabla}\psi \right\}$$

To derive a field equation, we can vary S with respect to the real and imaginary parts of ψ, or, equivalently, with respect to ψ and ψ^* separately.

$$\delta S = \int d^4x \left\{ \delta\psi^* \left(-i\hbar \frac{\partial}{\partial t} \psi\right) + \psi^* \, -i\hbar \frac{\partial}{\partial t} \delta\psi + \frac{1}{2}\frac{\hbar^2}{m} \vec{\nabla}\delta\psi^* \cdot \vec{\nabla}\psi + \frac{1}{2}\frac{\hbar^2}{m} \vec{\nabla}\psi^* \cdot \vec{\nabla}\delta\psi \right\}$$

after integ by parts

$$= \int d^4x \left\{ \delta\psi^* \left[-i\hbar \frac{\partial}{\partial t}\psi - \frac{1}{2}\frac{\hbar^2}{m} \nabla^2 \psi \right] + \left[i\hbar \frac{\partial}{\partial t} \psi^* - \frac{1}{2}\frac{\hbar^2}{m} \nabla^2\psi^* \right] \delta\psi \right\}$$

so that

$$\delta S = 0$$

implies the Schrödinger equation and its complex conjugate.

The Lagrangian density above is invariant under the transformation

$$\psi(x) \to e^{i\alpha}\psi(x) \qquad \psi^*(x) \to \psi^*(x)\, e^{-i\alpha}$$

with α a constant. However, it is not invariant to a local gauge transformat. If

$$\psi'(x) = e^{i\alpha(x)}\psi(x)$$

then $$\vec{\nabla}\psi'(x) = e^{i\alpha(x)}\left[\vec{\nabla}\psi(x) + i(\vec{\nabla}\alpha(x))\,\psi(x)\right]$$

To build a Lagrangian density that is invariant to this transformat, we need to introduce an additional vector field $\vec{a}$.

We write

$$\vec{D} = (\vec{\nabla} - i\vec{a}) \qquad \text{the "covariant derivative"}$$

and give $\vec{a}$ the transformat

$$\psi'(x) = e^{i\alpha(x)}\psi(x) \qquad \vec{a}'(x) = \vec{a}(x) + \vec{\nabla}\alpha$$

Then

$$(\vec{D}\psi)' = (\vec{\nabla} - i\vec{a}')\psi'$$

$$= (\vec{\nabla} - i\vec{a} - i\vec{\nabla}\alpha)\, e^{i\alpha(x)}\psi$$

$$= e^{i\alpha(x)}\left[\vec{\nabla} + i(\vec{\nabla}\alpha) - i\vec{a} - i(\vec{\nabla}\alpha)\right]\psi$$

$$= e^{i\alpha(x)}(\vec{D}\psi)$$

Then $$(\vec{D}\psi)^*\cdot(\vec{D}\psi) = (\vec{D}\psi)'^*\cdot(\vec{D}\psi)'$$

It is useful to make a a 4-vector and write

$$D_\mu = (\partial_\mu + i a_\mu)$$

then we have

$$-D^i = (-\nabla^i + i a^i) \qquad \text{(as before)}$$

$$D^0 = \left(\frac{1}{c}\frac{\partial}{\partial t} + i a^0\right) \qquad \text{so} \qquad \frac{\partial}{\partial t} \to c D^0$$

then

$$S = \int d^4x \left\{ \psi^* (-i\hbar c D^0)\psi + \frac{1}{2}\frac{\hbar^2}{m} (\vec{D}\psi)^* (\vec{D}\psi) \right\}$$

is locally gauge invariant.

Now we must write down some dynamical equation for a^μ. This equation should follow from a Lagrangian that is invariant to

$$\vec{a}' = \vec{a} + \vec{\nabla}\alpha$$

or, more generally, to

$$a'^\mu = a^\mu - \partial^\mu \alpha$$

The simplest way to insure ~~this~~ is to write the Lagrange density in terms of the antisymmetric combination

$$f^{\mu\nu} = \partial^\mu a^\nu - \partial^\nu a^\mu$$

thus, the simplest form is $\quad (\mathbb{C} = \text{constant})$

$$S = \int d^4x \left\{ -\tfrac{1}{4}\mathbb{C} \left(f^{\lambda\sigma} f_{\lambda\sigma} \right) \right\} + S_{\text{Schrödinger}}$$

Let's vary this first term with respect to a:

$$\delta S = \int d^4x \left\{ -\tfrac{1}{4}\mathbb{C} \cdot 2 \cdot f^{\lambda\sigma} (\partial_\lambda \delta a_\sigma - \partial_\sigma \delta a_\lambda) \right\}$$

$$= \int d^4x \left\{ -\tfrac{1}{4}\mathbb{C} \cdot 4 \cdot f^{\lambda\sigma} \partial_\lambda \delta a_\sigma \right\} \quad (\text{since } f^{\lambda\sigma} \text{ is antisymmetric})$$

integrate by parts $\quad = \int d^4x \left\{ \mathbb{C} \, \delta a_\sigma \, (\partial_\lambda f^{\lambda\sigma}) \right\}$

so if $\delta S = 0$ for an arbitrary variation of a, and we ignore matter,

$$\partial_\lambda f^{\lambda\sigma} = 0$$

which looks a lot like the first two Maxwell's equations.

We can make the connection clean in the following way:

Write:

$$\vec{a} = \frac{q}{\hbar} \vec{A}$$

where q is the electric charge of the Schrödinger particle.

Similarly

$$a^0 = \frac{q}{\hbar c} \phi$$

Then the equation above is just a constant times

$$\partial_\lambda F^{\lambda\sigma} = 0$$

If we choose $\mathbb{C}$ so that

$$S_A = \int d^4x \left\{ -\frac{1}{4} \frac{1}{c\mu_0} \left(F^{\lambda\sigma} F_{\lambda\sigma} \right) \right\}$$

Then the variation of this term together with the Schrödinger action gives

$$\partial_\lambda F^{\lambda\sigma} = \mu_0 J^\sigma$$

where (again, by Noether's theorem) J^σ is a conserved current. This is actually an equation for A^μ; the statement that F is written in terms of A gives the other two Maxwell equations.

The coupling of ϕ, $\vec{A}$ to a Schrödinger particle that follows from this analysis:

$$i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \left(\vec{\nabla} - i\frac{q}{\hbar}\vec{A} \right)^2 \psi + q\phi(x)\psi$$

is exactly that defended in the last lecture of 120.

The postulate of <u>local gauge invariance</u> does, then, provide the full structure of Maxwell's equations and the coupling of electromagnetic fields to matter.