# The Electromagnetic Potentials

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We now begin the general computation of the emission of electromagnetic radiation by accelerating or decelerating charges. I will approach this problem in two steps. First, I would like to reduce the problem of radiation in Maxwell's equations to a problem involving a scalar wave equation. Then we will solve the scalar problem, finding a very beautiful geometrical solution.

To simplify Maxwell's equations, we can do the analogue in the dynamic case of what we did in 120 for statics. We will introduce potentials whose derivatives are the $\vec{E}$ and $\vec{B}$ fields, observing that the potentials obey somewhat less complicated equations. Once we have solved the equations for the potentials, we can recover the corresponding $\vec{E}$ and $\vec{B}$ fields.

In electrostatics, we found potentials by solving the equations

$$\vec{\nabla}\times\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

In the dynamic case, $\vec{\nabla}\cdot\vec{B} = 0$ still holds, so it is still true that, in any topologically trivial region, $\vec{B}$ can be represented as

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

The equation for $\vec{\nabla} \times \vec{E}$ is now generalized to

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

but if $\vec{B} = \vec{\nabla} \times \vec{A}$, this is

$$\vec{\nabla} \times \left( \vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

so, in any topologically trivial region, $\vec{E}$ can be represented by

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} \phi$$

or

$$\vec{E} = -\vec{\nabla} \phi - \frac{\partial}{\partial t} \vec{A}$$

Then $\vec{E}, \vec{B}$ are described by potentials $(\phi, \vec{A})$.

If all is true and just, the 4-component object $(\phi, \vec{A})$ should be representable as a 4-vector. Indeed, write

$$A^{\mu} = (\phi/c, \vec{A})$$

Then we can build an antisymmetric tensor by the simple relation

$$F^{\prime\mu\nu} = c\,(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

The $\mu\nu = 12$ component of this equation is

$$F^{\prime 12} = -c\,B^3 = c\,(\partial^1 A^2 - \partial^2 A^1)$$

$$= c\,\left(-\frac{\partial}{\partial x^1} A^2 + \frac{\partial}{\partial x^2} A^1\right)$$

$$= -c\,(\vec{\nabla}\times\vec{A})^3$$

The $\mu\nu = 10$ component of this equation is

$$F^{\prime 10} = E^1 = c\,(\partial^1 A^0 - \partial^0 A^1)$$

$$= c\,\left(-\frac{\partial}{\partial x^1}\,\phi/c - \frac{1}{c}\frac{\partial}{\partial t} A^1\right)$$

$$= -\nabla^1\phi - \frac{\partial}{\partial t} A^1$$

as required. So we can refer questions about the tensor $F^{\mu\nu}$ back to questions about the 4-vector $A^\mu$.

In the static case, we saw that there are many choices for $\vec{A}$ that produce the same $\vec{B}$. $\vec{B}$ is unchanged by the shift

$$\vec{A} \to \vec{A} + \vec{\nabla}\lambda$$

which is called a gauge transformation. In the dynamic case, we see that this same freedom exists. If

$$A'^{\mu} = A^{\mu} + \partial^{\mu}\lambda$$

where $\lambda$ is any scalar function, $A$ and $A'$ define the same $\vec{E}$ and $\vec{B}$ fields. In components,

$$\phi' = \phi + \frac{\partial}{\partial t}\lambda$$

$$\vec{A}' = \vec{A} - \vec{\nabla}\lambda$$

(You can easily check that $\lambda$ cancels out of the formula for $\vec{E}$.) In practical problems, we may "specify a gauge" by putting an additional condition on $A^{\mu}$ that restricts this freedom. In 120, we used, in static problems, the condition

$$\vec{\nabla}\cdot\vec{A} = 0$$

This same condition is useful in some dynamic problems, it is called "Coulomb gauge" or "radiation gauge". An alternative choice is

$$\partial_{\mu}A^{\mu} = 0$$

This choice is called "Lorentz gauge", though it is actually due to Lorenz, a contemporary of Maxwell.

What do Maxwell's equations look like in terms of potentials?

Consider first the homogeneous Maxwell equations:

$$\epsilon^{\mu\nu\lambda\sigma} \partial_\nu F_{\lambda\sigma} = 0$$

Putting $F_{\lambda\sigma} = c(\partial_\lambda A_\sigma - \partial_\sigma A_\lambda)$, this is

$$\epsilon^{\mu\nu\lambda\sigma}[\partial_\nu \partial_\lambda A_\sigma - \partial_\nu \partial_\sigma A_\lambda] = 0$$

which is satisfied identically, since $\partial_\nu \partial_\lambda = \partial_\lambda \partial_\nu$.
(Actually, we derived the potentials by solving these equations.)
Putting the potential form into the inhomogeneous Maxwell equations:

$$\partial_\mu F^{\mu\nu} = \frac{1}{c\epsilon_0} J^\nu$$

we find

$$c\, \partial_\mu [\partial^\mu A^\nu - \partial^\nu A^\mu] = \frac{1}{c\epsilon_0} J^\nu$$

$$(\partial_\mu \partial^\mu) A^\nu - \partial^\nu (\partial_\mu A^\mu) = \frac{1}{c^2\epsilon_0} J^\nu$$

In Lorentz gauge, this equation becomes very simple:

Since $\partial_\mu A^\mu = 0$, we find

$$\Box A^\nu = \frac{1}{c^2\epsilon_0} J^\nu$$

where
$$\Box = \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

is the wave operator. This equation preserves the Lorentz gauge condition if $J^\nu$ is conserved: Taking $\partial_\nu$

$$\Box(\partial_\nu A^\nu) = \frac{1}{c^2\epsilon_0}\partial_\nu J^\nu = 0$$

Since we always deal with conserved currents, we have now reduced the problem of solving Maxwell's equations for a fixed current source to that of solving the scalar wave equation.

It is instructive to write the $\vec{E}$ and $\vec{B}$ fields of an electromagnetic wave in terms of potentials. Consider

$$\phi = 0$$
$$\vec{A}(x,t) = \mathrm{Re}\left\{ A_0 \hat{\epsilon}\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} \right\} \qquad \omega = c|\vec{k}|$$

If $\vec{k}\cdot\hat{\epsilon} = 0$, this satisfies both the conditions $\vec{\nabla}\cdot\vec{A} = 0$ and $\partial_\mu A^\mu = 0$. The corresponding $\vec{E}$ and $\vec{B}$ fields are:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A}$$

$$= -\frac{\partial}{\partial t}\vec{A} = \mathrm{Re}\left\{ i\omega A_0 \hat{\epsilon} \; e^{-i\omega t + i\vec{k}\cdot\vec{x}} \right\}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \mathrm{Re}\left\{ i\vec{k}\times\hat{\varepsilon} \, A_0 \; e^{-i\omega t + i\vec{k}\cdot\vec{x}} \right\}$$

Identifying $E_0 = i\omega A_0$ $\quad B_0 = i|\vec{k}|\, A_0 = \frac{E_0}{c}$,

we recognize this as the standard form of an electromagnetic wave.