Lorentz Transformation of Electromagnetic Field Configurations

Now that we have worked out the Lorentz transformation rules for electric and magnetic fields, let's apply these to some illustrative field configurations.

First of all, consider an electromagnetic wave localized in a wavepacket.

$$\vec{E}(\vec{r},t) = E_0 \hat{\epsilon} \cos(k(z-ct)) \, \mathcal{E}(\vec{r},t)$$

$$\vec{B}(\vec{r},t) = \frac{E_0}{c} \hat{z} \times \hat{\epsilon} \cos(k(z-ct)) \, \mathcal{E}(\vec{r},t)$$

where $\mathcal{E}(\vec{r},t)$ is the envelope function

$$\mathcal{E}(\vec{r},t) = \exp\left[-\frac{(z-ct)^2}{2a^2}\right] e^{-(x^2+y^2)/2a^2}$$

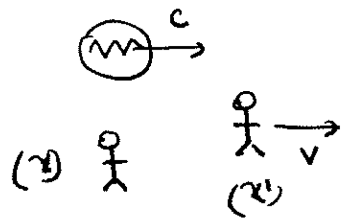
The volume occupied by the wave packet is

$$\mathcal{V} = \int d^3x \, \mathcal{E}^2 = [\sqrt{\pi}\, a]^{3/2}$$

In the limit $ka \gg 1$ or $a \gg \lambda$, the energy carried by the wavepacket is

$$\text{Energy} = \tfrac{1}{2}\epsilon_0 E_0^2 \cdot \mathcal{V}$$

If $\vec{E}, \vec{B}$ are the fields observed in the (x) frame, what fields are observed in the (x') frame moving with respect to (x) by the velocity $v\hat{3}$?



There are two parts to this conversion. First, we must transform the E and B fields into one another as described in the previous lecture. But, as you may recall from 121, we must also transform the place where the fields are evaluated. We showed then that, if $\phi(x)$ is a scalar field configuration measured by (x), (x') will see

$$\phi'(x) = \phi(\Lambda x)$$

This makes sense, because if $\phi(x)$ has a maximum at $x = a$, ϕ' has a maximum at a' s.t.

$$a = \Lambda a'$$

which is just right.

Using this transformation, the factor

$\cos k(z-ct)$ in $\vec{E}, \vec{B}$ becomes

$$\cos\left(k\left[\gamma(z+\beta ct) - \gamma(ct+\beta z)\right]\right)$$

$$= \cos\left(k\gamma(1-\beta)\,(z-ct)\right)$$

The wave seen by (x') still moves at speed c. But k and ω are transformed by a "Doppler shift"

$$k' = k\,\gamma(1-\beta) \qquad \omega' = \omega\,\gamma(1-\beta)$$

The conversion factor is

$$\gamma(1-\beta) = \frac{1-v/c}{\sqrt{1-v^2/c^2}} = \sqrt{\frac{1-v/c}{1+v/c}}$$

If the (x') observer is chasing the wave $(v>0)$, the frequency of the wave appears to <u>decrease</u> ("red shift"). If $v<0$, so that

then $\omega' > \omega$ and the wave gets a "blue shift". The formula for the transformation of k, ω has a nice symmetry between these cases.

It is worth contrasting this transformation law with that for sound waves or other waves that travel in a medium. Consider first the case in which the (x) frame is at rest with respect to the medium. The waveform is

$$\cos \; k(z - \bar{c}t)$$

where $\bar{c}$ is the speed of sound. Using the Gallilean boost

$$z = z' + vt' \qquad t = t'$$

an observing moving at v with respect to (x) (and the medium) sees.

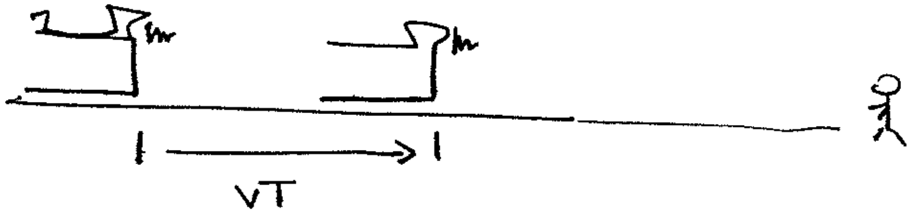
$$\cos(\; k((z+vt) - \bar{c}t))$$

$$= \cos \; k\,[z - (\bar{c}-v)t\,]$$

so the (x') observer sees the same wavelength $k' = k$, and the velocity $\bar{c}' = \bar{c} - v$. The frequency is shifted by

$$\omega' = \omega\,(1 - \frac{v}{\bar{c}})$$

There is a distinct situation in which the source of the waves is moving with respect to the medium while the observer is at rest with respect to the medium. This is the situation of a whistle on a moving train. If the train is at rest, its whistle may be thought of as an oscillator with period T, producing a sound wave of frequency $\omega = \frac{2\pi}{T}$ and wavelength $\lambda = \frac{2\pi}{k} = \frac{2\pi \bar{c}}{T}$. If the train is moving toward the observer.



the distance between successive crest is made shorter by VT, so the wavelength of the wave is now

$$\lambda' = \frac{2\pi}{T}(\bar{c} - V)$$

the wave number is $$k' = k \frac{\bar{c}}{\bar{c} - V}$$

and the frequency of this sound wave in air is $\omega' = \bar{c}k' = \omega \frac{\bar{c}}{\bar{c} - V}$

It the situation of a red shift, where the train is moving away from the observer

$$\omega' = \omega \frac{1}{(1 + V/\bar{c})}$$

To recap: source at rest w. respect to medium, observer moving : $\omega' = \omega(1 - V/\bar{c})$

observer at rest w. respect to medium, source moving : $\omega' = \omega \frac{1}{1 + V/c}$

In both cases $\omega' \cong \omega(1 - v/c + \cdots)$

but at higher orders in v/c the two cases are distinctly different. For a light wave in special relativity, there cannot be a difference that depends on whether the source or the observer is moving, and indeed the Doppler shift formula neatly splits the difference:

$$\omega' = \omega \sqrt{\frac{1 - v/c}{1 + v/c}} \quad \text{for a red shift.}$$

Now let's perform the rest of the transformation of the E and B fields. Since we go from (x) to (x'), the transformation is the reverse of p. 11 of the previous lecture. Since both $\vec{E}$ and $\vec{B}$ are transverse to the boost

$$\vec{E}_0' = \gamma(\vec{E}_0 + \vec{v} \times \vec{B}_0)$$

$$= \gamma(E_0 \hat{\epsilon} + v \hat{z} \times (\vec{z} \times \vec{\epsilon}) \frac{E_0}{c})$$

$$= \gamma(1 - v/c) \, E_0 \hat{\epsilon}$$

$$\vec{B}_0' = \gamma(\vec{B}_0 - \frac{v}{c^2} \hat{z} \times \vec{E}_0)$$

$$= \gamma(1 - v/c) \frac{E_0}{c} (\hat{z} \times \hat{\epsilon})$$

so the relation $E^2 = c^2 B^2$ holds in the (x') frame (as it must) and

$$E_0' = \sqrt{\frac{1-V/c}{1+V/c}}\; E_0$$

The energy density in the wave is then

$$\mathcal{E}' = \left(\frac{1-V/c}{1+V/c}\right) \mathcal{E}$$

Now, the envelope function $\mathcal{E} = e^{-(z-ct)^2/2a^2}$

transforms to

$$\mathcal{E}' = \exp\left[-\left[\gamma(1-\beta)(z-ct)\right]^2/2a^2\right]$$

so (x') actually sees the wave packet expanded by a factor

$$\frac{1}{\gamma(1-\beta)} = \sqrt{\frac{1+V/c}{1-V/c}}$$

In all, the energy of the packet transforms as

$$E' = \int d^3x \; \mathcal{E}' = \sqrt{\frac{1+V/c}{1-V/c}}\left(\frac{1-V/c}{1+V/c}\right) \cdot \int d^3x \; \mathcal{E}$$

or

$$E' = \left(\frac{1-V/c}{1+V/c}\right)^{1/2} E = \gamma(1-\beta) E$$

Similarly, using the momentum density $\vec{\mathcal{P}} = \epsilon_0 \vec{E} \times \vec{B}$, and $\vec{P} = \int d^3x \; \vec{\mathcal{P}}$, we see that

$$\vec{P}' = \left(\frac{1-v/c}{1+v/c}\right)^{1/2} \vec{P} = \gamma(1-\beta)\vec{P}$$

the relations for the original wave

$$E = \tfrac{1}{2}\epsilon_0 E_0^2 V \qquad \vec{P} = \tfrac{1}{2}\epsilon_0 E_0^2 \tfrac{1}{c} V \hat{z}$$

implying

$$E = c\vec{P}$$

this relation also holds according to (α'): $E' = c\vec{P}'$!

Now, we define the mass of a particle by

$$(mc)^2 = \left(\frac{E}{c}\right)^2 - (\vec{p})^2 = P^\mu P_\mu$$

Thus, a wave packet of electromagnetic radiation is also a <u>massless</u> <u>particle</u>, moving at the speed of light. The transformation of E and $\vec{p}$ for a particle moving in the $\hat{z}$ direction is:

$$\left(\frac{E'}{c}\right) = \gamma\left(\frac{E}{c} - \beta P\right)$$

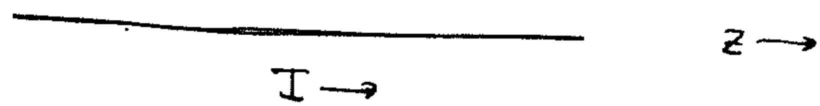
$$\text{(massless)} \qquad = \gamma(1-\beta)\frac{E}{c}$$

so

$$E' = \gamma(1-\beta)E \qquad P' = \gamma(1-\beta)P$$

in agreement with the above.

Next, consider the case of a wire directed along the $\hat{z}$ axis



In 121, we analyzed the force on a particle moving parallel to the wire with velocity v by transforming the current in the wire and using Maxwell's equations to find the fields in the (x') frame. We can also find that field directly using the Lorentz transformation of fields. In the (x) frame

$$\vec{B} = \frac{\mu_0}{2\pi r} I \, \hat{\phi} \qquad \vec{E} = 0$$

$\vec{B}$ is transverse to the boost. So in the (x') frame

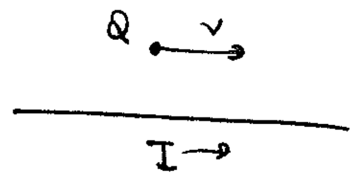
$$\vec{B}' = \gamma \left(\vec{B} - \frac{\vec{V}}{c^2} \vec{E} \right) = \gamma \vec{B}$$

$$\vec{E}' = \gamma \left(\vec{E} + \vec{V} \times \vec{B} \right) = \gamma \, \vec{V} \times \vec{B}$$

Again

$$\vec{B}' = \frac{\mu_0 \gamma}{2\pi r} I \, \hat{\phi} \qquad \vec{E}' = \frac{\mu_0 \gamma I}{2\pi r} \cdot v \cdot (-\hat{r})$$

So in the (x') frame there is an $\vec{E}$ field pointing radially into the wire. A particle moving at $v\hat{z}$ in the (x) frame



is at rest in the (x') frame. Thus it feels the electrical force

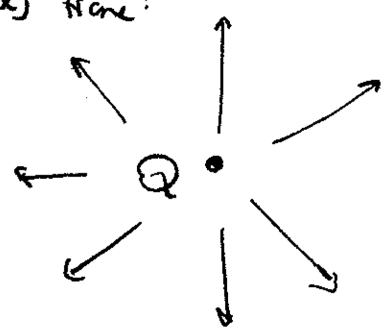
$$\vec{F}' = Q\frac{\mu_0 \gamma I}{2\pi r}\cdot v\,(-\hat{r})$$

Transforming back to the (x) frame, since $\vec{F}'$ is transverse to the boost

$$\vec{F} = Q v \frac{\mu_0 I}{2\pi r}(-\hat{r})$$

as required.

Finally, consider the case of an isolated chge Q at rest in the (x) frame:



In the (x) frame, the fields are:

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} = \frac{Q}{4\pi\epsilon_0}\frac{1}{[x^2+y^2+z^2]^{3/2}}(x,y,z)$$

$$\vec{B} = 0$$

In the (x') frame, we must first evaluate this solution at a Lorentz transformed point

$$x \to x \qquad y \to y \qquad z \to \gamma(z+vt),$$

then transform the fields so that

$$E'^3 = E^3$$

$$B'^3 = B^3 = 0$$

$$\vec{E}'_\perp = \gamma(\vec{E}_\perp + \vec{v}\times\vec{B})_\perp = \gamma\vec{E}_\perp$$

$$\vec{B}'_\perp = \gamma\left(\vec{B}_\perp - \frac{\vec{v}}{c^2}\times\vec{E}_\perp\right) = -\gamma\,\frac{\vec{v}\times\vec{E}_\perp}{c^2}$$

The new E field is then:

$$\vec{E}' = \frac{Q}{4\pi\epsilon_0}\,\frac{1}{[x^2+y^2+\gamma^2(z+vt)^2]^{3/2}}\,(\gamma x, \gamma y, \gamma(z+vt))$$

If $z_0(t) = -vt\,\hat{z}$

$$\vec{E}' = \frac{Q}{4\pi\epsilon_0}\,\gamma\,\frac{1}{[x^2+y^2+\gamma^2(z-z_0(t))^2]^{3/2}}\,(x, y, z-z_0(t))$$

The new B field is

$$\vec{B}' = -\frac{\vec{v}}{c^2}\times\vec{E}' = \frac{Q\gamma}{4\pi\epsilon_0}\,\frac{v/c^2\,(-y, x)}{[x^2+y^2+\gamma^2(z-z_0(t))^2]^{3/2}}$$

So, with respect to (x'), the particle moves along a trajectory

$$x = 0 \quad y = 0 \qquad z = z_0(t) = -vt\,\hat{z}$$

The $\vec{E}$ field points radially outward from the instantaneous location of the charge

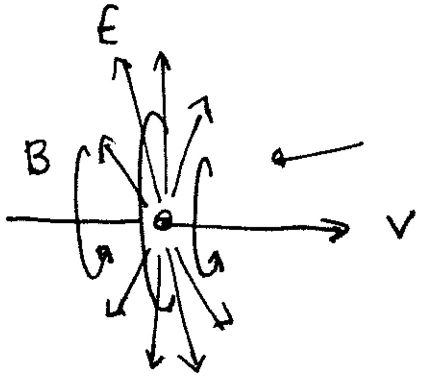
$$\hat{r} = \text{unit vector} \parallel \text{to} \quad (x, y, z - z_0(t))$$

$$\vec{E}' = \frac{Q}{4\pi\epsilon_0} \frac{\gamma\,[x^2+y^2+(z-z_0)^2]^{1/2}}{[x^2+y^2+\gamma^2(z-z_0)^2]^{3/2}}\,\hat{r}$$

and $\vec{B}'$ points in the $(-\hat{\phi})$ direction

$$\vec{B}' = \frac{Q}{4\pi\epsilon_0} \frac{v}{c^2} \frac{\gamma\,[x^2+y^2]^{1/2}}{[x^2+y^2+\gamma^2(z-z_0)^2]^{3/2}}\,(-\hat{\phi})$$

If we send $\vec{v} \to -\vec{v}$ to get a charge moving in the $(+\hat{z})$ direction, we get the following picture:

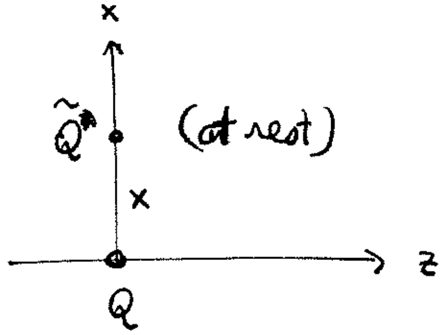


B field in the $+\hat{\phi}$ direction

$$\vec{B} = \frac{\vec{v}}{c^2} \times \vec{E}$$

E field contracted by γ in the direction of motion

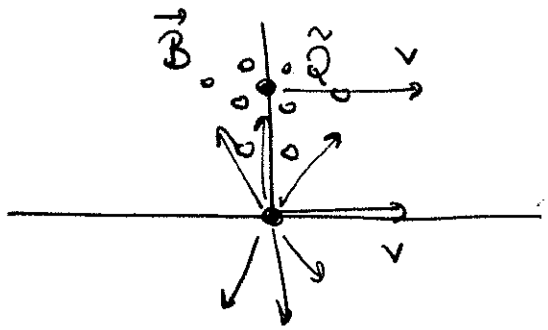
It is instructive to compare the force that a charge Q exerts on another charge $\tilde{Q}$ with respect to different observers. Let (x) see the situation:



$\tilde{Q}$ feels the force

$$\vec{F} = \frac{Q\tilde{Q}}{4\pi\epsilon_0} \frac{1}{x^2} \hat{x}$$

(x') sees:



$\tilde{Q}$ feels the electric force

$$\vec{F}'_E = \frac{Q\tilde{Q}}{4\pi\epsilon_0} \gamma \frac{1}{x^2} \hat{x}$$

and the magnetic force $\qquad \vec{v} \times \vec{B}'$

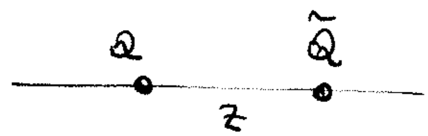
$$\vec{F}'_B = \tilde{Q} \cdot v \cdot \frac{Q}{4\pi\epsilon_0} \frac{\gamma v}{c^2} \frac{1}{x^2} (-\hat{x})$$

The total force is

$$\vec{F}' = \frac{Q\tilde{Q}}{4\pi\epsilon_0}\frac{1}{x^2}\,\hat{x}\;\;\gamma\left(1-v^2/c^2\right)$$

$$= \frac{1}{\gamma}\,\vec{F}\,.$$

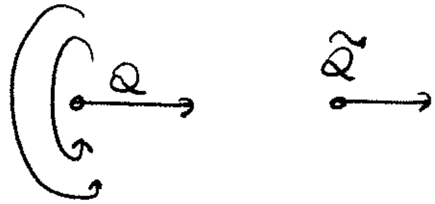
which is correct, because the force is transverse to the boost and the particle $\tilde{Q}$ is at rest in the frame (x). If in the (x) frame the charges are arranged as follows:



then

$$\vec{F} = \frac{Q\tilde{Q}}{4\pi\epsilon_0}\frac{1}{z^2}\,\hat{z}$$

In the (x') frame, there is no magnetic field on the z axis



so $\vec{F}_B' = 0$. The electric force is

$$\vec{F}' = \frac{Q\tilde{Q}}{4\pi\epsilon_0}\,\frac{\gamma}{\gamma^3}\,\frac{1}{(z')^2}\,\hat{z}$$

where (z') is the separation in the (x') frame. The distances z' and z are related by Lorentz contraction

$$z' = \sqrt{1 - V^2/c^2}\; z = \frac{1}{\gamma} z$$

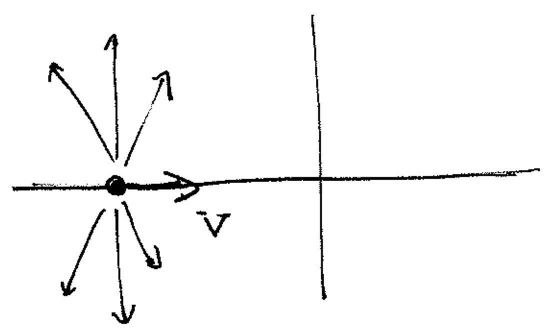
So in all

$$\vec{F}' = \frac{Q\tilde{Q}}{4\pi\epsilon_0} \frac{1}{z^2} \hat{z} = \vec{F}$$

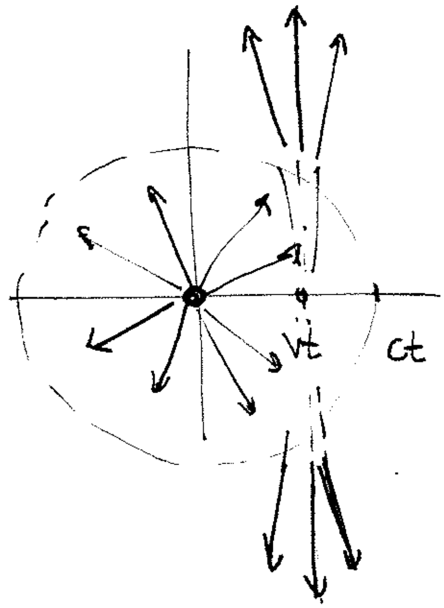
as required for a force parallel to the boost.

We have derived the formulae for the $\vec{E}$ and $\vec{B}$ fields of a moving charge assuming that the charge is in eternal uniform motion. What if it changes its velocity. Let's try to draw the picture of the fields of a particle that comes in from the left at the velocity $V\hat{z}$ and then stops at $z = 0$. Before it stops, it has the E and B fields that we have just described. The $\vec{E}$ fields point back to the position of the particle at

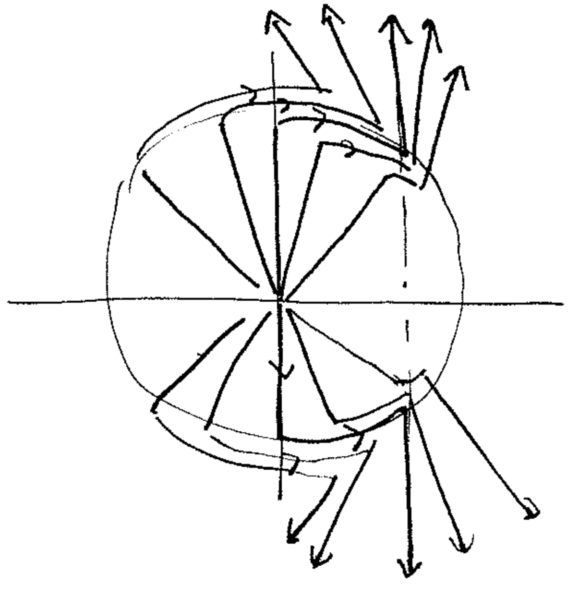
$z = Vt \quad (t < 0)$



After the charge stops, it should have a radial $\vec{E}$ field ~~centered~~ on $z = 0$. However, if a signal cannot propagate faster than the speed c, the fields at $r > ct$ will still be those of the moving charge:



There are no sources on the sphere $r = ct$, so $\vec{\nabla}\cdot\vec{E} = 0$, $\vec{\nabla}\cdot\vec{B} = 0$ there, and so the lines of $\vec{E}$ cannot end. Instead, they must join up:



To make the joint, there must be a transverse $\vec{E}$ field moving outward in the sphere $r \cong ct$. In other words, the stopping charge emits radiation! In the next few lectures, we will see how to compute this radiation field.

Before leaving our discussion of relativity, I would like to present the relativistic version of the formulae for energy-momentum conservation of the electromagnetic field. When we discussed the electric current, we saw that the charge density ρ and the current $\vec{j}$ fit together into a 4-vector J^μ:

$$J^\mu = (\rho c, \vec{j})$$

The relation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

takes the manifestly Lorentz-invariant form $\partial_\mu J^\mu = 0$.

The ~~integral~~ $\int d^3x$ of this equation is

$$\frac{d}{dt} Q = 0 \qquad \text{where} \qquad Q = \int d^3x \ \frac{1}{c} J^0$$

It is important to note that it follows only from the 4-vector character of J^μ and the conservation law that Q is Lorentz-invariant. Let's compare the values of the integral in the (x) frame and the (x') frame:

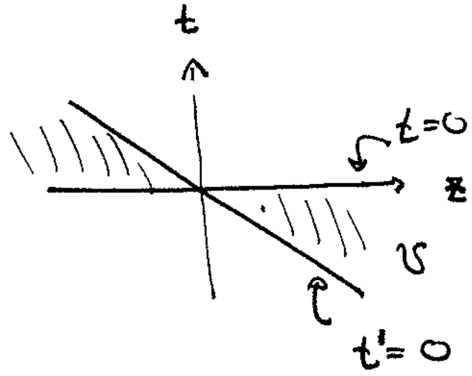
$$\int d^3x' \, J'^0 - \int d^3x \, J^0$$

$$= \int d^3x' \, n'_\mu J^\mu - \int d^3x \, n_\mu J^\mu$$

where n_μ is the normal to the surface of constant time:

$$= -\int_V d^4x \, \partial_\mu J^\mu$$

Where V is the volume enclosed by the surfaces $t=0$, $t'=0$



$$= 0 \;!$$

Now, what is the corresponding story for energy and momentum? In Maxwell's theory, energy and momentum are integrals of local densities

$$E = \int d^3x \, \mathcal{E} \qquad \vec{P} = \int d^3x \, \vec{\mathcal{P}}$$

To make $(E/c, \vec{P}) = P^\mu$ a 4-vector, we need a structure

$$P^\mu = \int d^3x \, T^{0\mu}$$

where $T^{\lambda\mu}$ is a tensor with two covariant 4-vector indices. This equation is, more explicitly

$$\mu=0 \qquad \frac{E}{c} = \int d^3x \, T^{00}$$

$$\mu=i \qquad P^i = \int d^3x \, T^{0i}$$

so $T^{00} = \dfrac{\mathcal{E}}{c}$ $\qquad T^{0i} = \mathcal{P}^i$

A natural form for the conservation law obeyed by $T^{\lambda\mu}$ is

$$\partial_\lambda T^{\lambda\mu} = 0$$

The $\mu=0$ equation is

$$\frac{1}{c}\frac{\partial}{\partial t} T^{00} + \nabla^i T^{i0} = 0$$

so we identify $T^{i0} = \frac{1}{c^2} j_{\mathcal{E}}^i = \frac{1}{c^2} S^i$ in electromagnetism. Setting $\mu = i$ gives the equations

$$\frac{1}{c}\frac{\partial}{\partial t} \mathcal{T}^{0i} + \nabla_j \mathcal{T}^{ji} = 0$$

so $\mathcal{T}^{ji} = \frac{1}{c} T^{ji}$. Remarkably, the single object $\mathcal{T}^{\lambda\mu}$ contains all of the energy and momentum densities and fluxes:

$$\begin{array}{ccc} & \mu = 0 & j \\ \mathcal{T}^{\lambda\mu} = \begin{array}{c} \lambda = 0 \\ i \end{array} & \left(\begin{array}{c|c} \mathcal{E}/c & p^j \\ \hline j^i_{\mathcal{E}}/c^2 & T^{ij}/c \end{array} \right) & \end{array}$$

Last term we argued that T^{ij} is a symmetric matrix. If this is also true for $\mathcal{T}^{\lambda\mu}$, then there is a relation

$$p^i = j^i_{\mathcal{E}}/c^2$$

In electromagnetism we found:

$$\vec{j}_{\mathcal{E}} = \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$\vec{p} = \epsilon_0 \vec{E} \times \vec{B}$$ so this is right!

Now we ought to be able to write a single, grand, relativistically covariant expression for all of the energy and momentum densities and fluxes. It is not hard to find this expression — it must be quadratic in $F^{\mu\nu}$ and symmetric, with two free Lorentz indices. Here it is:

$$\mathcal{T}^{\mu\nu} = -\frac{\epsilon_0}{c}\left[F^{\mu\lambda} F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu} F^{\lambda\sigma} F_{\lambda\sigma}\right]$$

Recall that

$$E^i = F^{i0} \qquad F^{ij} = -c\,\epsilon^{ijk} B^k$$

and

$$-\frac{1}{2} F^{\lambda\sigma} F_{\lambda\sigma} = E^2 - c^2 B^2$$

then

$$\mathcal{T}^{00} = -\frac{\epsilon_0}{c}\left[F^{0i} F^{0}{}_{i} + \frac{1}{2}\overbrace{\eta^{00}}^{=1}\left[E^2 - c^2 B^2\right]\right]$$

$$= -\frac{\epsilon_0}{c}\left[-E^2 + \frac{1}{2}E^2 - \frac{1}{2}\frac{1}{\epsilon_0\mu_0}B^2\right]$$

or

$$\mathcal{T}^{00} = \frac{1}{c}\left\{\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right\}$$

$$= \mathcal{E}/c \quad \checkmark$$

$$\mathcal{T}^{0i} = -\frac{\epsilon_0}{c} F^{0j} F^{i}{}_{j} \qquad \text{since } \eta^{0i} = 0$$

$$= -\frac{\epsilon_0}{c} (-E^j)\, c\, \epsilon^{ijk} B^k$$

$$= \epsilon_0 (\vec{E} \times \vec{B})^i = \mathcal{P}^i = \frac{j_\mathcal{E}^i}{c^2} \quad \checkmark$$

$$\mathcal{T}^{ij} = -\frac{\epsilon_0}{c} \left\{ F^{ia} F^{j}{}_{a} - \frac{1}{2}\delta^{ij}\left(E^2 - \frac{1}{\epsilon_0\mu_0} B^2\right) \right\}$$

$$= -\frac{\epsilon_0}{c} \left\{ F^{i0} F^{j}{}_{0} + F^{ik} F^{j}{}_{k} - \frac{1}{2}\delta^{ij} E^2 + \frac{\delta^{ij}}{2}\frac{1}{\epsilon_0\mu_0} B^2 \right\}$$

$$= -\frac{\epsilon_0}{c} \left\{ E^i E^j + c^2 \epsilon^{ikl} B^l \epsilon^{jkm} B^m - \frac{1}{2}\delta^{ij} E^2 + \frac{\delta^{ij}}{2}\frac{1}{\epsilon_0\mu_0} B^2 \right\}$$

$$= -\frac{1}{c} \left\{ \epsilon_0 E^i E^j - c^2 \delta^{ij} B^2 + c^2 B^i B^j - \frac{1}{2}\delta^{ij} E^2 + \frac{1}{2}\delta^{ij} c^2 B^2 \right\}$$

$$= \frac{1}{c} \left\{ \left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2\right)\delta^{ij} - \epsilon_0 E^i E^j - \frac{1}{\mu_0} B^i B^j \right\}$$

$$= T^{ij} \quad \checkmark$$