Synchrotron Radiation

Another important application of radiation from extremely relativistic particles is <u>synchrotron radiation</u>. We learned earlier in the course that a charged particle moves in a circle or helix in a constant magnetic field. Such a particle is constantly accelerated, and so it constantly radiates. This radiation appears in many contexts. In astrophysics, for example, a pulsar is a collapsed star that carries a large magnetic field. Synchrotron radiation from electrons trapped in the field is the source of the <u>pulse</u>.

Recently, however, synchrotron radiation has acquired an important scientific role as the preferred method for constructing bright sources of X-rays. Put electrons into a ring of magnets and accelerate them to GeV energies, and, as we will see, they radiate X-rays. Such X-ray sources are orders of magnitude brighter than standard discharge tubes and allow control of the X-rays' energy and polarization. This allows beautiful applications in solid-state physics, chemistry, and molecular biology. Many of these applications were pioneered at the storage ring SPEAR at SLAC, which has operated since the early 1970's.

To begin, let's review the motion of relativistic particles in a magnetic field. The equation of motion is

$$\frac{d\vec{p}}{dt} = q\vec{v}\times\vec{B}$$

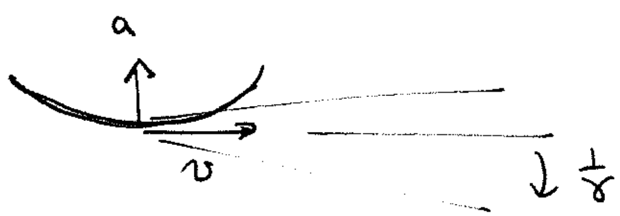
For a particle moving in a circle at angular frequency

$$\Omega = \frac{v}{r} \qquad v = \text{velocity} \quad r = \text{radius of circle.}$$

$$\Omega p = qvB$$

now $p = mv\gamma$, so $\Omega = \frac{qB}{m\gamma}$, generalizing the cyclotron frequency $\omega_c = \frac{qB}{m}$ found at nonrelativistic velocities.

What does the radiation from such a charge look like? Recall ~~that~~, for a particle accelerating in the $\hat{x}$ direction and moving rapidly in the $\hat{y}$ direction

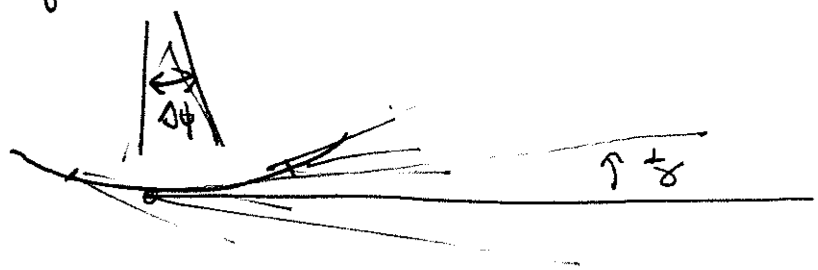


the radiation is boosted forward into a cone of size $\Theta \sim \frac{1}{\gamma}$. As $v \to c$, this cone becomes very narrow.

In this lecture, I will be ~~interested~~ in electrons ($mc^2 = 0.5\,\text{MeV}$)

accelerated by GeV energies, so $\gamma \sim$ few $\times 10^3$. Then we can approximate $v/c = \beta \approx 1$ where appropriate, while $\gamma \gg 1$.

Before working out the distribution of synchrotron radiation ~~quantitatively~~, I would like to discuss its major qualitative feature. For a detector placed at some point away from the circle, the radiation coming to that detector will come from a small arc of the circle near the intersection of the tangent



The size of the arc is $\Delta\phi \sim \frac{1}{\gamma}$ or $\Delta l = \frac{r}{\gamma}$. Radiation emitted from the front of this arc takes a time $\Delta t_\ell = \frac{r}{c\gamma}$ to cross the arc. But the electrons take a very similar time $\Delta t_e = \frac{r}{v\gamma}$. So, the light pulse that is radiated in the direction of the detector has length

$$\Delta t_e - \Delta t_\ell = \frac{r}{\gamma}\left(\frac{1}{v} - \frac{1}{c}\right)$$

$$= \frac{r}{\gamma c}\left(\frac{1}{\beta} - 1\right)$$

now
$$\frac{1}{\beta} - 1 = \frac{1-\beta}{\beta} = \frac{1-\beta^2}{\beta(1+\beta)} \approx \frac{1}{2\gamma^2}$$

if we set $\beta \to 1$ where appropriate. Thus the pulse of radiation is only as long as

$$\Delta t = \frac{r}{2c} \frac{1}{\gamma^3}$$

The Fourier decomposition of the pulse must then contain all frequencies up to

$$\omega_c \sim \frac{c}{r} \gamma^3 = \Omega \gamma^3$$

This enhancement by γ^3 is remarkable and is what allows synchrotron radiation to produce very high frequencies. For example SPEAR contains 3 GeV electrons:

$$\gamma = \frac{E}{mc^2} = 6 \times 10^3$$

The radius is about 100 m, so $\Omega = \frac{c}{r} \cong 3 \times 10^6/\text{sec}$.
Confining electrons to such a path requires a magnetic field

$$B = \frac{m\gamma\Omega}{q} \cong 0.1\,T = 1 \text{ kgauss}$$

which is quite reasonable. And,

$$\omega_c \sim \Omega\gamma^3 \sim 7 \times 10^{17}/\text{sec}$$

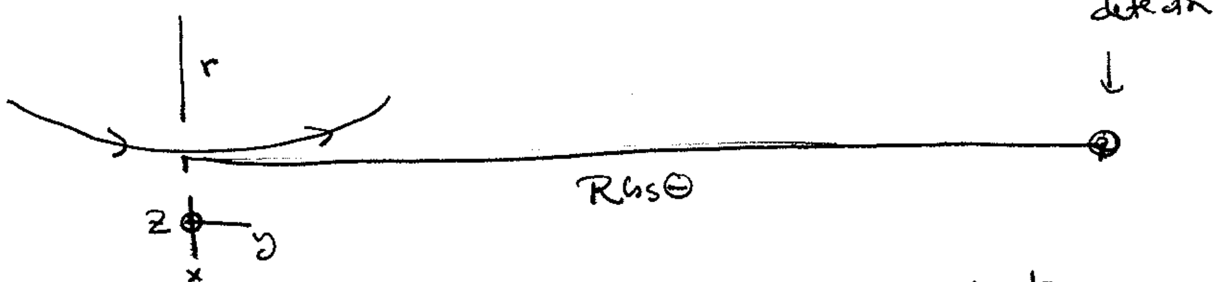
This corresponds to $\hbar\omega_c \sim 0.4$ keV or $\lambda = \frac{2\pi c}{\omega} \sim 30\,\mathring{A}$, rather hard X-rays.

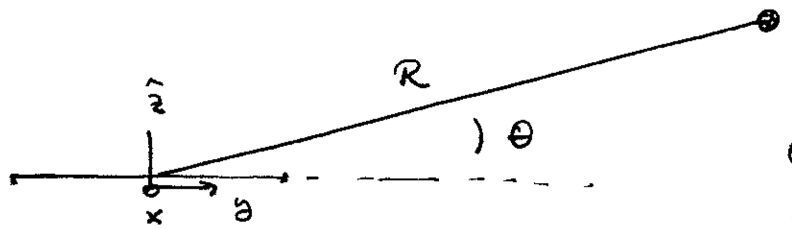
Now let's compute the distribution of synchrotron radiation more carefully. Our starting point is the formula from our study of the Lienard-Wiechert potentials:

$$\vec{E}(t,x) = \frac{e}{4\pi\epsilon_0}\left[\frac{1}{(R_* - \vec{\beta}_*\cdot\vec{R})^3}\left\{(1-\beta_*^2)\vec{\mathcal{R}} + \frac{1}{c^2}\vec{R}_*\times(\vec{\mathcal{R}}\times\vec{a})\right\}\right]$$

where $\mathcal{R} = \vec{R}_* - \vec{\beta}_* R$, the expression in [ ] to be evaluated at the time of emission.

Set up the geometry:





Θ = angle of rise out of the plane of the orbit

Let the particle's motion be

$$\vec{r}(t) = (r\cos\Omega t,\ r\sin\Omega t,\ 0) \qquad \Omega = \frac{v}{r}$$

$$\vec{\beta}(t) = \frac{\dot{\vec{r}}}{c} \qquad \frac{\vec{a}}{c} = \dot{\vec{\beta}}$$

In the expression above, we can replace

$$R_* \to R$$

and drop the $1-\beta_*^2$ term, since $\beta_* \to 1$

then

$$\vec{E}(t,\vec{x}) = \frac{e}{4\pi\epsilon_0}\frac{1}{Rc}\left[\frac{1}{(1-\vec{\beta}\cdot\hat{R})^3}\quad \hat{R}\times(\hat{R}-\vec{\beta})\times\dot{\vec{\beta}}\right]_{\text{time of emission}}$$

with $\quad \hat{R} = (0, \cos\Theta, \sin\Theta)$

I would like to continue the calculation in frequency space. Let's compute

$$\vec{E}(\omega,\vec{x}) = \int_{-\infty}^{\infty} dt\; e^{i\omega t}\; \vec{E}(t,\vec{x})$$

Then the power radiated will be proportional to $\int \frac{d\omega}{2\pi} |E(\omega,\vec{x})|^2$

In fact

$$\frac{dP}{d\Omega} = R^2 \cdot \frac{1}{2\mu_0 c} \int \frac{d\omega}{2\pi} |\vec{E}(\omega,\vec{x})|^2$$

so $|\vec{E}(\omega,\vec{x})|^2$ will give the frequency spectrum of the radiation.

now

$$\vec{E}(\omega,\vec{x}) = \int_{-\infty}^{\infty} dt\; e^{i\omega t}\; \frac{e}{4\pi\epsilon_0}\frac{1}{Rc}\frac{1}{(1-\vec{\beta}(t_*)\cdot\hat{R})^3}\,\hat{R}\times(\hat{R}-\vec{\beta}(t_*))\times\dot{\vec{\beta}}(t_*)$$

where the time of emission t_* is related to the detection time t by

$$t = \left(\frac{R}{c} - \frac{\hat{R}\cdot\vec{r}(t_*)}{c}\right) + t_*$$

or $$t = \frac{R}{c} + t_* - \frac{\hat{R}\cdot\vec{r}(t_*)}{c}$$

In
$$e^{i\omega t} = \underbrace{e^{i\omega R/c}} \quad e^{i\omega(t_* - \hat{R}\cdot\frac{\vec{r}(t_*)}{c})}$$

the first factor is a pure phase which squares to 1; I will drop it from here on. We also need

$$dt = dt_* - \hat{R}\cdot\frac{\dot{\vec{r}}}{c}\,dt_*$$

or
$$dt = (1-\hat{R}\cdot\vec{\beta}(t_*))\,dt_*$$

then
$$\vec{E}(\omega,\vec{x}) = \frac{e}{4\pi\epsilon_0 RC}\int_{-\infty}^{\infty} dt_*\, e^{i\omega(t_* - \hat{R}\cdot\vec{r}(t_*)/c)}$$

$$\cdot \frac{1}{(1-\hat{R}\cdot\vec{\beta}(t_*))^2}\,\hat{R}\times(\hat{R}-\vec{\beta}(t_*))\times\dot{\vec{\beta}}(t_*)$$

Next, notice ~~that~~

$$\frac{d}{dt}\left(\frac{\hat{R}\times(\hat{R}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{R}}\right) = \frac{d}{dt}\left(\frac{\hat{R}\,\hat{R}\cdot\vec{\beta} - \vec{\beta}}{1-\vec{\beta}\cdot\hat{R}}\right)$$

$$= \frac{(-\dot{\vec{\beta}} + \hat{R}\,\hat{R}\cdot\dot{\vec{\beta}})(1-\vec{\beta}\cdot\hat{R}) + (-\vec{\beta}+\hat{R}\,\hat{R}\cdot\vec{\beta})\,\dot{\vec{\beta}}\,\hat{R}}{(1-\vec{\beta}\cdot\hat{R})^2}$$

$$= \frac{-\dot{\vec{\beta}} + \hat{R}\,\hat{R}\cdot\dot{\vec{\beta}} + \dot{\vec{\beta}}\,\vec{\beta}\cdot\vec{R} - \vec{\beta}\,\dot{\vec{\beta}}\cdot\hat{R} - \cancel{\hat{R}\,\hat{R}\cdot\dot{\vec{\beta}}\,\hat{R}\cdot\vec{\beta}} + \cancel{\hat{R}\,\hat{R}\cdot\vec{\beta}\,\hat{R}\cdot\dot{\vec{\beta}}}}{(1-\vec{\beta}\cdot\hat{R})^2}$$

$$= \frac{\hat{R}\times(\hat{R}\times\dot{\vec{\beta}}) - \hat{R}\times(\vec{\beta}\times\dot{\vec{\beta}})}{(1-\vec{\beta}\cdot\hat{R})^2}$$

which is just the second line of the integrand! So

$$\vec{E}(\omega,\vec{x}) = \frac{e}{4\pi\epsilon_0 R c}\int_{-\infty}^{\infty} dt_* \; e^{i\omega(t_* - \vec{r}\cdot\hat{R}/c)} \cdot \frac{d}{dt_*}\left(\frac{\hat{R}\times\hat{R}\times\vec{\beta}}{1-\vec{\beta}\cdot\hat{R}}\right)_*$$

integrate by parts!

$$= \frac{e}{4\pi\epsilon_0 R c}\int dt_* \; (-i\omega)\,\cancel{(1-\hat{R}\cdot\vec{\beta}(t_*))}\; \frac{\hat{R}\times\hat{R}\times\vec{\beta}}{\cancel{1-\vec{\beta}\cdot\hat{R}}}(t_*)$$

$$\cdot\; e^{i\omega(t_* - \vec{r}\cdot\hat{R}/c)}$$

$$= \frac{-i\omega e}{4\pi\epsilon_0 R c}\int_{-\infty}^{\infty} dt_* \; e^{i\omega(t_* - \vec{r}\cdot\hat{R}/c)}\left(\hat{R}\times\underline{\hat{R}\times\vec{\beta}(t_*)}\right)$$

This expression has now become very simple. From here on, I will write $\underline{t}$ for $\underline{t_*}$ in the integral.

To evaluate the integral, we need the explicit expressions for $\vec{r}(t)$ and $\hat{R}$.

$$\vec{r}(t) = r(\cos\Omega t,\ \sin\Omega t,\ 0)$$

$$\vec{\beta}(t) = \frac{v}{c}(-\sin\Omega t,\ \cos\Omega t,\ 0)$$

$$\hat{R} = (\ 0\ ,\ \cos\Theta,\ \sin\Theta\ )$$

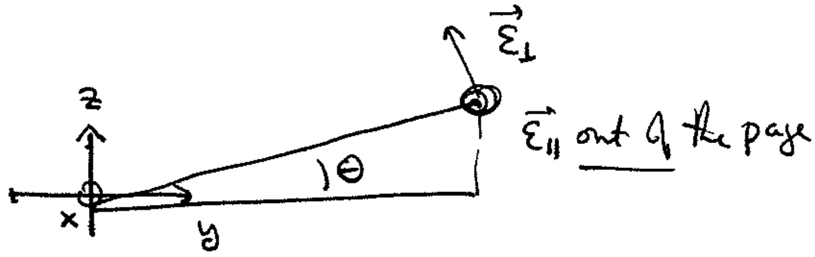
so $\hat{R}\times\vec{\beta} = \frac{v}{c}\,(-\cos\Omega t\sin\Theta\ ,\ -\sin\Omega t\sin\Theta\ ,\ +\sin\Omega t\cos\Theta)$

$$\hat{R}\times(\hat{R}\times\vec{\beta}) = \left(\frac{v}{c}\right)(\ \sin\Omega t\ ,\ -\cos\Omega t\sin^2\Theta\ ,\ \cos\Omega t\sin\Theta\cos\Theta)$$

we can set this = 1

so if we set up unit vectors

$$\vec{\epsilon}_{\parallel} = \hat{x} \qquad \vec{\epsilon}_{\perp} = (\,0\,, -\sin\theta, \cos\theta)$$



$$\hat{R} \times (\hat{R} \times \vec{\beta}) = \sin\Omega t \; \vec{\epsilon}_{\parallel} + \cos\Omega t \; \sin\theta \; \vec{\epsilon}_{\perp}$$

polarization $\parallel$ to $\hat{x}$ — polarization out of the plane of the orbit.

So far, we have not used the fact that the radiation is boosted forward, i.e. that the radiation comes from only a very small arc of the circular orbit. To take advantage of this, make the approximations:

$$\Omega t \ll 1 \qquad \theta \ll 1 \qquad \beta \simeq 1$$

$$\hat{R} \times (\hat{R} \times \vec{\beta}) \cong \Omega t \; \vec{\epsilon}_{\parallel} + \theta \; \vec{\epsilon}_{\perp}$$

In the exponent, we must be more careful:

$$t - \hat{R}/c \cdot \vec{r}(t) = t - \frac{r}{c}\cos\theta \sin\Omega t$$

$$= t - \frac{r}{c}\left(\Omega t - \tfrac{1}{6}(\Omega t)^3 + \dots\right)\left(1 - \tfrac{1}{2}\theta^2 + \dots\right).$$

Now $\frac{r}{c}\Omega = \frac{v}{c} \approx 1$ so $t - \frac{r\Omega}{c} t = (1-\beta)t$

which is very small. So we must keep the first correction to this term:

$$t - \frac{\hat{R}}{c}\vec{r}(t) \cong (1-\beta)t + \frac{1}{6}\Omega^2 t^3 + \frac{1}{2}\frac{r}{c}\theta^2 \Omega t$$

$$(1-\beta) = \frac{1-\beta^2}{1+\beta} \cong \frac{1}{2\gamma^2}$$

so

$$\cong \frac{1}{2}\left(\frac{1}{\gamma^2} + \theta^2\right)t + \frac{1}{6}\Omega^2 t^3 + \cdots$$

Finally, then

$$\vec{E}(\omega,\vec{x}) = \frac{-i\omega e}{4\pi\epsilon_0 Rc}\int_{-\infty}^{\infty} dt\, e^{i\omega\left[\frac{1}{2}\left(\frac{1}{\gamma^2}+\theta^2\right)t + \frac{1}{6}\Omega^2 t^3\right]}\left(\Omega t\,\vec{\epsilon}_{\parallel} + \theta\,\vec{\epsilon}_{\perp}\right)$$

Does this integral look familiar? Let $u = \left(\frac{1}{2}\omega\Omega^2\right)^{1/3} t$,

$$z = \frac{1}{2}\omega\left(\frac{1}{\gamma^2}+\theta^2\right) \Big/ \left(\frac{1}{2}\omega\Omega^2\right)^{1/3} = \left(\frac{\omega}{2\Omega}\right)^{2/3}\left(\frac{1}{\gamma^2}+\theta^2\right)$$

and we have

$$\vec{E}(\omega,\vec{x}) = \frac{-i\omega e}{4\pi\epsilon_0 Rc}\cdot\left(\frac{2}{\omega\Omega^2}\right)^{1/3}$$

$$\cdot\int_{-\infty}^{\infty} du\, e^{iuz + i\frac{u^3}{3}}\left(\left(\frac{2\Omega}{\omega}\right)^{1/3} u\,\vec{\epsilon}_{\parallel} + \theta\,\vec{\epsilon}_{\perp}\right)$$

This is our friend, the Airy function! We are in the region $z > 0$, where $Ai(z) \to 0$ exponentially as $z \to \infty$

$$\vec{E}(\omega,\vec{x}) = \frac{-i\omega e}{4\pi\epsilon_0 R c}\left(\frac{2}{\omega\Omega^2}\right)^{1/3}$$

$$\cdot 2\pi \cdot \left\{ \theta \cdot Ai(z)\ \vec{\epsilon}_\perp + \left(\frac{2\Omega}{\omega}\right)^{1/3}\left(-i\frac{dAi(z)}{dz}\right)\vec{\epsilon}_\parallel \right\}$$

Using

$$Ai(z) = \frac{\sqrt{z}}{\pi\sqrt{3}} K_{1/3}\left(\frac{2}{3}z^{3/2}\right)$$

$$Ai'(z) = -\frac{z}{\pi\sqrt{3}} K_{2/3}\left(\frac{2}{3}z^{3/2}\right)$$

$$\sqrt{z} = \left(\frac{\omega}{2\Omega}\right)^{1/3}\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}$$

$$\frac{2}{3}z^{3/2} = \frac{2}{3}\frac{\omega}{2\Omega}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}$$

$$\vec{E}(\omega,\vec{x}) = \frac{-i\omega e}{4\pi\epsilon_0 R c}\frac{1}{\omega}\frac{1}{\pi\sqrt{3}}\left[\frac{1}{\gamma^2}+\theta^2\right]^{1/2}\cdot 2\pi$$

$$\cdot\left(\theta \cdot K_{1/3}\left(\frac{1}{3}\frac{\omega}{\Omega}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}\right)\vec{\epsilon}_\perp\right.$$

$$\left. + i\left[\frac{1}{\gamma^2}+\theta^2\right]^{1/2} K_{2/3}\left(\frac{1}{3}\frac{\omega}{\Omega}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}\right)\vec{\epsilon}_\parallel\right)$$

collecting terms and forming $\frac{dP}{d\Omega}$:

$$\frac{dP}{d\omega\, d\Omega} = \frac{e^2}{48\pi^3 \epsilon_0} \frac{\omega^2}{\Omega^2} \left[\frac{1}{\gamma^2} + \theta^2\right]^2$$

$$\cdot \left[\underbrace{\left(\frac{\theta^2}{(\frac{1}{\gamma})^2 + \theta^2}\right) K^2_{1/3}(\xi)}_{\text{polarizn } \parallel \text{ to } \vec{\varepsilon}_\perp} + \underbrace{K^2_{2/3}(\xi)}_{\text{polarizing } \parallel \text{ to } \vec{\varepsilon}_\parallel} \right]$$

Let's examine the dependence of this formula on ω and θ.

The variable ξ is

$$\xi = \frac{1}{3} \frac{\omega}{\Omega} \left[\frac{1}{\gamma^2} + \theta^2\right]^{3/2}$$

Look first at $\theta = 0$. Then the synchrotron radiation is completely polarized parallel to $\hat{x}$. For this case

$$\xi = \frac{1}{3} \frac{\omega}{\Omega} \frac{1}{\gamma^3} = \omega / \omega_c$$

where $\omega_c = 3 \Omega \gamma^3$

as we estimated on p. 4. Since $K_\nu(\xi) \sim e^{-\xi}$,

the synchrotron radiation spectrum cuts off as

$$\sim e^{-2\omega/\omega_c}$$

But ω_c is unexpectedly large, so there is ample radiation at very high frequencies!

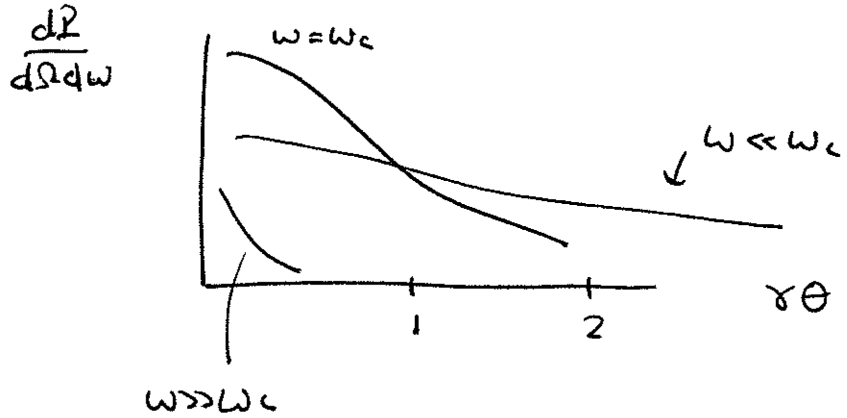
Now look at the angular dependence. For very large angles ($\theta \gg \frac{1}{\gamma}$)

$$\xi = \frac{1}{3}\frac{\omega}{\Omega}\,\theta^{3/2}$$

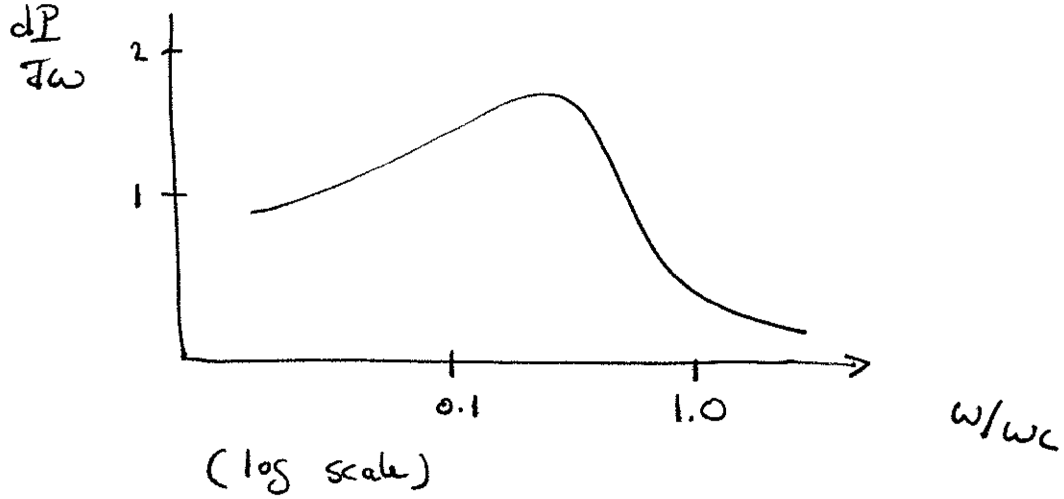
so the angular cutoff takes place at

$$\theta_c = \left(3\frac{\Omega}{\omega}\right)^{1/3} = \frac{1}{\gamma}\left(\frac{\omega_c}{\omega}\right)^{1/3}$$

thus, high frequencies are strongly cut off, but low frequencies are radiated out to much larger θ



The spectrum integrated over angles has the form:



So by positioning a detector carefully, we can have radiation of a controlled energy and polarization, with energy of order $\omega_c = 3\Omega\gamma^3$!