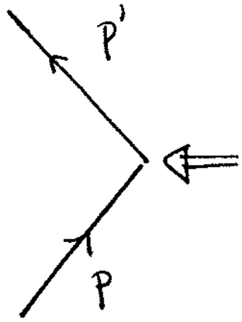
Radiation from Extremely Relativistic Particles

May 30

Most of our discussion of radiation has been for charged particles in nonrelativistic motion. I would now like to present some examples of radiation from charged particles moving at speeds very close to the speed of light. We discussed the radiation from relativistic particles earlier by boosting the radiation from an accelerating particle instantaneously at rest. In that discussion, we were mainly concerned with the spatial distribution of radiation. However, in applications that involve high-energy particles, it is typically more important to understand the distribution of radiation in $\omega, \vec{k}$, since the radiation is emitted as quanta (photons) with energy $E = \hbar\omega$ and momentum $\vec{p} = \hbar\vec{k}$.

Let's begin, then, by analyzing a very simple situation with relativistic particles and computing the radiation pattern in $(\omega, \vec{k})$. The situation I would like to consider is that in which a particle comes in with momentum p^μ, experiences a sudden kick at $\vec{x} = 0$, $t = 0$, and exits with a different momentum p'^μ. Thus:



sudden kick applied at $t=0,\ \vec{x}=0$

[The sudden transition from p to p' is a bit unphysical; we'll correct for this later.]

we can describe the particle's trajectory in space-time as

$$y^\mu(\tau) = \begin{cases} \dfrac{p^\mu}{m}\tau & \tau < 0 \\ \dfrac{p'^\mu}{m}\tau & \tau \end{cases}$$

Notice that, from $\tau = 0$ to $\tau = \tau$

$$(y^\mu(\tau) - y^\mu(0))^2 = \frac{p'^\mu p'_\mu}{m^2}\tau^2 = \frac{(mc)^2}{m^2}\tau^2 = (c\tau)^2$$

so τ is indeed proper time. Now I would like to write the <u>current</u> associated with this motion. Earlier in the course, we wrote, for a particle moving on a trajectory $(t_y(\sigma), \vec{y}(\sigma))$

$$J^\mu(x) = q\int d\sigma\ \frac{dy^\mu}{d\sigma}\ \delta(t_x - t_y(\sigma))\ \delta^{(3)}(\vec{x} - \vec{y}(\sigma))$$

Evaluating with $\sigma = t_y$, this is just

$$J^0 = qc\ \delta(x - \vec{y}(t_x)) \qquad \vec{J} = q\vec{V}\ \delta^{(3)}(x - \vec{y}(t_x))$$

where $\vec{V}(t_x) = \left.\dfrac{d\vec{y}}{dt}\right|_{t=t_x}$. For this application, it will be helpful to write J^μ in a way that allows us to use Lorentz invariance as much as possible. In particular, I would now like to use $x^0 = ct$ rather than t. Define

$$\delta^{(4)}(x-y) = \delta(x^0-y^0)\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$= \delta(ct_x - ct_y)\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$= \frac{1}{c}\,\delta(t_x-t_y)\,\delta^{(3)}(\vec{x}-\vec{y})$$

so

$$J^\mu = q\int d\sigma \;\frac{dy^\mu}{d\sigma}\cdot c\cdot\; \delta^{(4)}(x-y(\sigma))$$

and choose $\sigma = \tau$ (proper time) [and set $q=e$, the elementary charge]. Then

$$J^\mu = e\int d\tau\;\frac{dy^\mu}{d\tau}\cdot c\cdot\;\delta^{(4)}(x-y(\tau))$$

For our example, $y(\tau)$ is given on p. 2.

Now let's solve for the $A^\mu(x)$ set up by this current:

$$\Box A^\mu = \mu_0 J^\mu \qquad\qquad \Box = \frac{\partial^2}{\partial x^{0\,2}} - \nabla^2$$

we can solve this equation by going to Fourier space. Write

$$A^\mu(x) = \int\frac{d^4k}{(2\pi)^4}\; e^{-ik\cdot x}\;\tilde{A}^\mu(k)$$

note that $k\cdot x = k^0x^0 - \vec{k}\cdot\vec{x}$, and if we set $k^0 = \frac{\omega}{c}$,

$$k\cdot x = k^0 x^0 - \vec{k}\cdot\vec{x} = \omega t - \vec{k}\cdot\vec{x} \quad \text{as usual.}$$

Then
$$\Box\, e^{-ik\cdot x} = (-ik^\mu)^2 e^{-ik\cdot x} = -k^2 e^{-ik\cdot x}$$

where $k^2 = (k^0)^2 - |\vec{k}|^2$. Thus

$$-k^2 \tilde{A}^\mu(k) = \mu_0 \tilde{J}^\mu(k)$$

We need to compute $\tilde{J}^\mu(k)$:

$$\tilde{J}^\mu(k) = \int d^4x \; e^{ik\cdot x} \; J^\mu(x)$$

$$= \int d^4x \; e^{ik\cdot x} \left\{ \int_0^\infty d\tau \; e\frac{p'^\mu}{m}c \; \delta^{(4)}\!\left(x - \frac{p'}{m}\tau\right) \right.$$
$$\left. + \int_{-\infty}^0 d\tau \; e\frac{p^\mu}{m}c \; \delta^{(4)}\!\left(x - \frac{p}{m}\tau\right) \right\}$$

$$= ec \cdot \left\{ \int_0^\infty d\tau \; \frac{p'^\mu}{m} \; e^{ik\cdot\frac{p'}{m}\tau} + \int_{-\infty}^0 d\tau \; \frac{p^\mu}{m} e^{ik\frac{p}{m}\tau} \right\}$$

The two integrals here have oscillating integrands, so we should treat them carefully. Mathematically, these integrals are convergent only with an appropriate analytic continuation. Physically, we should define the integrals so that we can ignore the contribution from the very far past or far future. So, for the first integral, define it as

$$\lim_{\epsilon\to 0} \int_0^\infty d\tau \; p'^\mu \, e^{ik\cdot\frac{p'}{m}\tau} \, e^{-\epsilon\tau}$$

The value of this integral is

$$\frac{p'^\mu}{m}\,\frac{1}{\epsilon - i\frac{k\cdot p'}{m}} = \frac{i\,p'^\mu}{k\cdot p' + i\varepsilon}$$

Similarly, the second integral should be written:

$$\lim_{\epsilon\to 0}\int_{-\infty}^{0} d\tau\, \frac{p^\mu}{m}\, e^{ik\cdot\frac{p}{m}\tau}\, e^{\epsilon\tau} = \frac{p^\mu}{m}\,\frac{1}{\epsilon + i\frac{k\cdot p}{m}}$$

$$= -i\,\frac{p^\mu}{k\cdot p - i\varepsilon}$$

so

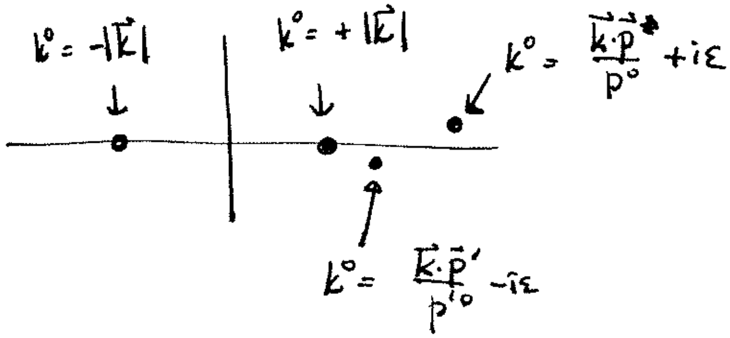
$$\tilde{A}^\mu(k) = -\frac{\mu_0}{k^2}\, iec\left[\frac{p'^\mu}{k\cdot p' + i\varepsilon} - \frac{p^\mu}{k\cdot p - i\varepsilon}\right]$$

Now invert the Fourier transform and reconstruct $A^\mu(x)$:

$$A^\mu(x) = -iec\mu_0 \int\frac{dk^0}{2\pi}\int\frac{d^3k}{(2\pi)^3}\, e^{-ik\cdot x}\left[\frac{p'^\mu}{k\cdot p' + i\varepsilon} - \frac{p^\mu}{k\cdot p - i\varepsilon}\right]\frac{1}{k^2}$$

with $k\cdot p = k^0p^0 - \vec{k}\cdot\vec{p}$ $\quad k\cdot p' = k^0p'^0 - \vec{k}\cdot\vec{p}\,'$ $\quad k^2 = (k^0)^2 - |k|^2$

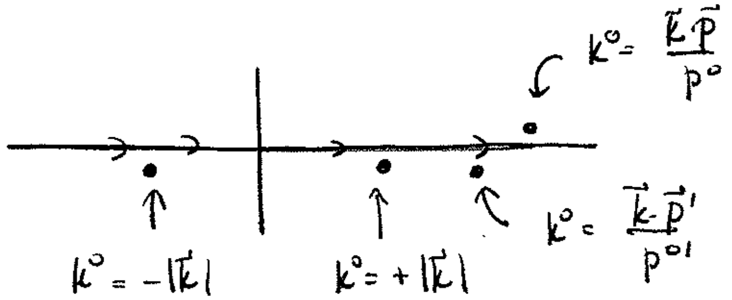
This is a complicated integral that has 4 poles in the complex k^0 plane:



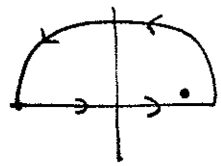
If we compute $A^\mu(x)$ using the retarded Green's function, the contour in the k^0 plane should go above the two poles at $k^0 = \pm|\vec{k}|$.

Thus:

$$\int \frac{dk^0}{2\pi} \quad e^{-ik^0x^0}$$



For $x^0 < 0$, close upward:



and pick up the pole at $k^0 = \frac{\vec{k}\cdot\vec{p}}{p^0}$. Then

$$A^\mu(x) = -iec\mu_0 \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{2\pi i}{2\pi} \left(-\frac{p^\mu}{p^0}\right)\left(\frac{1}{k^2}\right)\Bigg|_{k^0 = \frac{\vec{k}\cdot\vec{p}}{p^0}}$$

$$= \frac{e}{\epsilon_0 c} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{p^\mu}{p^0} \frac{1}{|\vec{k}|^2 - (k^0)^2}$$

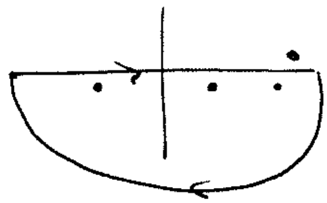
using $\mu_0\epsilon_0 = \frac{1}{c^2}$. Let's evaluate this in the rest frame of the initial particle: $p^\mu = (mc, \vec{0})$. Then $k^0 = \frac{\vec{k}\cdot\vec{p}}{p^0} = 0$ and

$$A^0 = \frac{e}{\epsilon_0 c} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{1}{|\vec{k}|} = \frac{1}{c} \cdot \frac{e}{4\pi\epsilon_0 |\vec{x}|}$$

$$\vec{A} = 0$$

These are just the potentials of the Coulomb field carried by the initial particle (as we should have expected!). In any other frame, we obtain the boost of this Coulomb field.

Now consider $x^0 > 0$. Here we close downward and pick up three poles



The pole at $k^0 = \frac{\vec{k}\cdot\vec{p}'}{p^{0\prime}}$ gives the Coulomb field of the outgoing particle. The two poles at $k^0 = \pm|\vec{k}|$ represent the radiation field. So let's evaluate just this contribution. From the pole at $k^0 = +|\vec{k}|$

$$A^\mu(x) = -iec\mu_0 \int \frac{d^3k}{(2\pi)^3} e^{-ik^0x^0} e^{i\vec{k}\cdot\vec{x}} \left(\frac{p^{\mu\prime}}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right) \frac{(-2\pi i)}{2k^0} \Bigg|_{k^0 = +|\vec{k}|}$$

$$= -\frac{e}{2\epsilon_0 c} \int \frac{d^3k}{(2\pi)^3} e^{-ik^0x^0} e^{i\vec{k}\cdot\vec{x}} \frac{1}{|\vec{k}|} \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right) \Bigg|_{k^0 = +|\vec{k}|}$$

The pole at $k^0 = -|\vec{k}|$ gives just the complex conjugate of this, so the total radiation field is

$$A^\mu_{rad}(x) = -\frac{e}{\epsilon_0 c} \mathrm{Re}\left\{ \int \frac{d^3k}{(2\pi)^3} e^{-ik^0x^0} e^{i\vec{k}\cdot\vec{x}} \frac{1}{|\vec{k}|} \left(\frac{p'^\mu}{k\cdot p} - \frac{p^\mu}{k\cdot p} \right) \right\}$$

where, from now on, $k^0 = +|\vec{k}|$.

From this expression, we can compute the $\vec{E}$ and $\vec{B}$ fields of the radiation. Let's write

$$\vec{E}(x) = \mathrm{Re}\left[\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, \tilde{\vec{E}}(k)\right] e^{-ik^0x^0}$$

$$\vec{B}(x) = \mathrm{Re}\left[\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, \tilde{\vec{B}}(k)\right] e^{-ik^0x^0}$$

then

$$\tilde{\vec{E}}(k) = -ic\vec{k}\,\tilde{A}^0(k) + ick^0\,\tilde{\vec{A}}(k)$$

Break this into pieces $\parallel$ and $\perp$ to $\vec{k}$:

$$= +ic|\vec{k}|\left[\tilde{\vec{A}}(k)\right]_\perp + \left(ic|\vec{k}|\,\tilde{\vec{A}}(k) - ic\vec{k}\,\tilde{A}^0(k)\right)_{\parallel \text{ to } \vec{k}}$$

In fact, the second term is zero:

$$(\;\;) = ic\vec{k}\cdot\tilde{\vec{A}}(k) - i|\vec{k}|c\tilde{A}^0(k)$$

$$= -i\frac{e}{\epsilon_0 c}\frac{c}{|\vec{k}|}\left\{\left(\frac{\vec{k}\cdot\vec{p}\,'}{k\cdot p'} - \frac{\vec{k}\cdot\vec{p}}{k\cdot p}\right) - \left(\frac{k^0p^{0\prime}}{k\cdot p'} - \frac{k^0p^0}{k\cdot p}\right)\right\}$$

$$= -i\frac{e}{\epsilon_0}\frac{1}{|\vec{k}|}\left\{-\frac{k\cdot p'}{k\cdot p'} + \frac{k\cdot p}{k\cdot p}\right\} = 0$$

$$\tilde{\vec{E}}(k) = +i|\vec{k}|\,\tilde{\vec{A}}(k) = -i\frac{e}{\epsilon_0}\left(\frac{\vec{p}\,'}{k\cdot p'} - \frac{\vec{p}}{k\cdot p}\right)_{\perp \text{ to } \vec{k}}$$

Let $\hat{\varepsilon}_{\perp i}$ be the two polarization unit vectors perpendicular to $\hat{k}$

$$\hat{\varepsilon}_{\perp j}\cdot\hat{k} = 0 \quad , \quad |\vec{\varepsilon}_{\perp j}|^2 = 1$$

then

$$\tilde{E}(k) = \sum_j -i\frac{e}{\epsilon_0}\, \hat{\varepsilon}_{\perp j}\, \hat{\varepsilon}_{\perp j}\cdot\left(\frac{\vec{p}\,'}{k\cdot p'} - \frac{\vec{p}}{k\cdot p}\right)$$

and

$$\tilde{B}(k) = \frac{\hat{k}}{c}\times\tilde{\vec{E}}(k) = -i\frac{e}{\epsilon_0 c}\,\hat{k}\times\left(\frac{\vec{p}\,'}{k\cdot p'} - \frac{\vec{p}}{k\cdot p}\right)$$

Now we can compute the energy that goes into the radiation field

$$\text{Energy} = \int d^3x\left(\frac{\epsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2\right)$$

write $$\vec{E} = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\left[e^{-ik\cdot x}\tilde{\vec{E}}(k) + e^{ik\cdot x}\tilde{E}^*(k)\right]$$

$$\int d^3x\,\frac{\epsilon_0}{2}E^2 = \int d^3x\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\,\frac{\epsilon_0}{8}\left[e^{-ik\cdot x}\tilde{E}(k) + e^{ik\cdot x}\tilde{E}^*(k)\right]\left[e^{-ik'\cdot x}E(k') + e^{+ik'\cdot x}\tilde{E}^*(k')\right]$$

using $$\int d^3x\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = (2\pi)^3\delta^{(3)}(k-k')$$

$$= \frac{\epsilon_0}{8}\int\frac{d^3k}{(2\pi)^3}\; 2\,\tilde{\vec{E}}(k)\cdot\tilde{\vec{E}}^*(k) + \text{terms w. } e^{\pm 2ik^0x^0}$$

$$\left\langle\int d^3x\,\frac{\epsilon_0}{2}E^2\right\rangle = \frac{\epsilon_0}{4}\int\frac{d^3k}{(2\pi)^3}\,\tilde{E}(k)\cdot\tilde{E}^*(k)$$

the magnetic field term contributes an equal amount, and we find:

$$\text{Energy} = \frac{\epsilon_0}{2} \int \frac{d^3k}{(2\pi)^3} \left| \vec{\tilde{E}}(k) \right|^2$$

$$= \frac{e^2}{2\epsilon_0} \int \frac{d^3k}{(2\pi)^3} \sum_j \left| \hat{\varepsilon}_{\perp j} \cdot \left(\frac{\vec{p}\,'}{k\cdot p'} - \frac{\vec{p}}{k\cdot p} \right) \right|^2$$

$$= \frac{4\pi}{2} \alpha \hbar c \int \frac{d^3k}{(2\pi)^3} \sum_j \left| \hat{\varepsilon}_{\perp j} \cdot \left(\frac{\vec{p}\,'}{k\cdot p'} - \frac{\vec{p}}{k\cdot p} \right) \right|^2$$

where I have introduced the dimensionless fine structure constant

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} = \frac{1}{137}.$$

It would be very nice to cast this equation into a more obviously Lorentz-covariant form. This can be done using a trick due to Feynman: Consider for definiteness the case in which $\hat{k} = \hat{z}$ then

$$\hat{k} = (0,0,1) \qquad \hat{\varepsilon}_{\perp 1} = (1,0,0) \qquad \hat{\varepsilon}_{\perp 2} = (0,1,0)$$

Now notice ~~that~~

$$k_\mu \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right) = \left(\frac{k\cdot p'}{k\cdot p'} - \frac{k\cdot p}{k\cdot p} \right) = 0$$

This is just the expression that the charge current is conserved

$$\partial_\mu J^\mu = 0$$

But now put $k^\mu = (|k|, 0, 0, |k|)$ and write this relation out explicitly:

$$k^0\left(\frac{p'^0}{k\cdot p'} - \frac{p^0}{k\cdot p}\right) - k\left(\frac{p'^3}{k\cdot p'} - \frac{p^3}{k\cdot p}\right) = 0$$

and, since $k^0 = k$, the two terms in parentheses are equal.

So

$$\sum_{j=1,2}\left|\hat{\varepsilon}_{\perp j}\cdot\left(\frac{\vec{p}'}{k\cdot p'} - \frac{\vec{p}}{k\cdot p}\right)\right|^2$$

$$= \sum_{j=1,2}\left|\hat{\varepsilon}_{\perp j}\left(\frac{\vec{p}'}{k\cdot p'} - \frac{\vec{p}}{k\cdot p}\right)\right|^2 + \left|\frac{p'^3}{k\cdot p'} - \frac{p^3}{k\cdot p}\right|^2 - \left|\frac{p'^0}{k\cdot p'} - \frac{p^0}{k\cdot p}\right|^2$$

$$= -\left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p}\right)^2 \quad \text{the Lorentz vector squared!}$$

$$= -\left(\frac{(p')^2}{(k\cdot p')^2} + \frac{p^2}{(k\cdot p)^2}\right) + 2\,\frac{p\cdot p'}{k\cdot p\;k\cdot p'}$$

so

$$\text{Energy} = \frac{4\pi\alpha}{2}\,\hbar c\int\frac{d^3k}{(2\pi)^3}\left[2\,\frac{p\cdot p'}{k\cdot p\;k\cdot p'} - \frac{(mc)^2}{(k\cdot p)^2} - \frac{(mc)^2}{(k\cdot p')^2}\right]$$

$$= \frac{4\pi\alpha\,\hbar c}{4\pi^2}\int_0^\infty dk\,k^2\int\frac{d\Omega}{4\pi}\left[2\,\frac{p\cdot p'}{k\cdot p\;k\cdot p'} - \frac{(mc)^2}{(k\cdot p)^2} - \frac{(mc)^2}{(k\cdot p')^2}\right]$$

Let's evaluate this formula for very relativistic particles going in and out

$$p^\mu = \frac{E}{c}(1, \vec\beta) \qquad p'^\mu = \frac{E'}{c}(1, \vec\beta')$$

with $\frac{E}{c} \gg mc$, $|\vec\beta| \approx |\vec\beta'| \approx 1$. In this case, the last two terms are $O(\frac{m^2c^4}{E^2})$ and we may neglect them.

$$p \cdot p' = \frac{EE'}{c^2}(1 - \vec\beta\cdot\vec\beta') \qquad k \cdot p = k\frac{E}{c}(1 - \hat{k}\cdot\vec\beta) \quad \text{etc.},$$

so
$$\text{Energy} = \frac{2}{\pi}\alpha\hbar c \int_0^\infty dk\, k^2 \int \frac{d\Omega}{4\pi} \frac{(1-\vec\beta\cdot\vec\beta')}{k^2(1-\hat{k}\cdot\vec\beta)(1-\hat{k}\cdot\vec\beta')}$$

$$= \frac{2}{\pi}\alpha\hbar c \int_0^\infty dk \int \frac{d\Omega}{4\pi} \frac{(1-\vec\beta\cdot\vec\beta')}{(1-\hat{k}\cdot\vec\beta)(1-\hat{k}\cdot\vec\beta')}$$

The integral over k formally goes up to ∞, but this is an artifact of our assumption that the particle bounces instantaneously from p^μ to p'^μ. If this process takes a finite time t, then the integral will run only up to $\omega = 1/t$ or $k = 1/ct$. And, there is one more important consideration. The energy is emitted in photons of energy $\hbar\omega = \hbar c k$. When this energy becomes of the order of E, E', we must take into account how the electromagnetic field extracts energy from the particles. Then these formulae will be modified.

The integral over $d\Omega$ is strongly peaked about the directions

$\hat{k} \parallel \vec{\beta}$ and $\hat{k} \parallel \vec{\beta}'$. Near $\hat{k} \parallel \vec{\beta}$, we can approximate:

$$1 - \hat{k}\cdot\vec{\beta}' \approx 1 - \hat{\beta}\cdot\vec{\beta}' \approx 1 - \vec{\beta}\cdot\vec{\beta}'$$

$$\int_{\hat{k} \text{ near } \hat{\beta}} \frac{d\Omega}{4\pi} \frac{(1-\vec{\beta}\cdot\beta')}{(1-\hat{k}\cdot\vec{\beta})(1-\hat{k}\cdot\vec{\beta}')} \cong \int^{1} \frac{d\cos\Theta}{2} \frac{1}{(1-\beta\cos\Theta)}$$

where $\hat{k}\cdot\hat{\beta} = \cos\Theta$.

$$\cong \frac{1}{2}\log\left(\frac{\Theta(1)}{1-\beta}\right)$$

and $\frac{1}{(1-\beta)} = \frac{(1+\beta)}{(1-\beta)(1+\beta)} \approx \frac{2}{1-\beta^2} = 2\gamma^2 = 2\frac{E^2}{(mc^2)^2}$

$$\simeq \frac{1}{2}\log\frac{E^2}{(mc^2)^2}$$

The region near $\hat{k} \parallel \vec{\beta}'$ gives a similar contribution. So

$$\text{Energy} \cong \frac{1}{\pi}\alpha\hbar c \int_0^E dk \left\{ \underbrace{\log\frac{E^2}{(mc^2)^2}}_{\hat{k}\parallel\vec{\beta}} + \underbrace{\log\left(\frac{E^2}{(mc^2)^2}\right)}_{\hat{k}\parallel\vec{\beta}'} \right\}$$

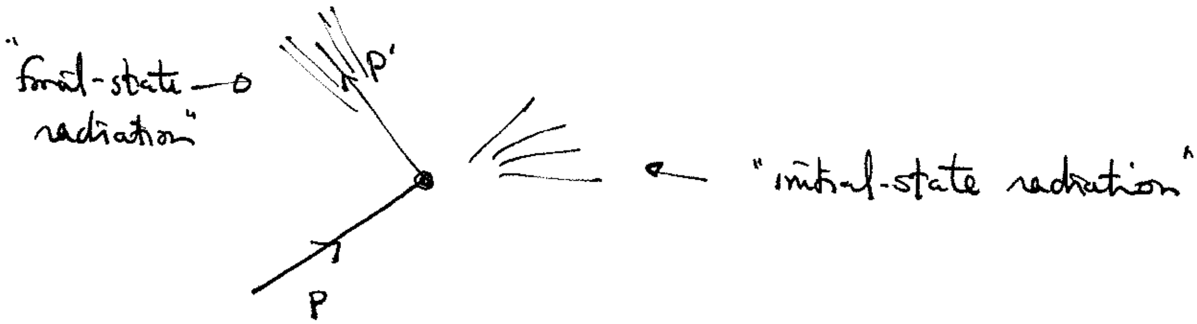
The radiated energy appears as photons of energy $E = \hbar\omega = \hbar c k$.

The number of photons is given by dividing by $\hbar c k$. So

$$N_\gamma = \frac{\alpha}{\pi}\int_0^E \frac{dk}{k}\left(\log\frac{E^2}{(mc^2)^2} + \log\frac{E^2}{(mc^2)^2}\right)$$

Notice that, as $k \to 0$, an infinite number of photons are radiated. But there is nothing wrong with this, since an infinite number of very low-energy $(k \to 0)$ photons can still carry only finite energy.

Most of the photons are radiated either along the direction of the incoming particle $\vec{p}$ or $\vec{\beta}$ or along the direction of the outgoing particle $\vec{p}\,'$ or $\vec{\beta}\,'$



The initial state radiation can be characterized as a density of photons at "longitudinal fraction" $x = \frac{kc}{E}$

$$N_\gamma \, (\parallel \text{ to } \vec{\beta}) \cong \int_0^1 dx \left(\frac{\alpha}{\pi} \frac{1}{x} \log \frac{E^2}{mc^2} \right)$$

This formula is only valid for $x \ll 1$, when we can ignore the effect of the radiation on the energy of the incoming particle. To take this into account properly, we need to go to quantum electrodynamics. There, one finds the Weiszäcker - Williams distribution

$$N_\gamma \, (\parallel \text{ to } \vec{\beta}) \cong \int_0^1 dx \left(\frac{\alpha}{2\pi} \frac{1}{x} \left[1 + (1-x)^2 \right] \log \frac{E^2}{(mc^2)^2} \right)$$

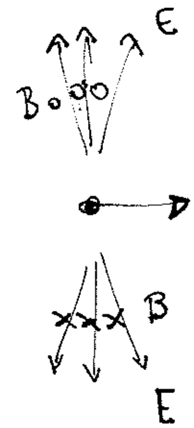
$$= \int_0^1 dx \quad f_\gamma(x)$$

similarly, the distribution of final-state radiation photons parallel to $\vec{\beta}'$ is

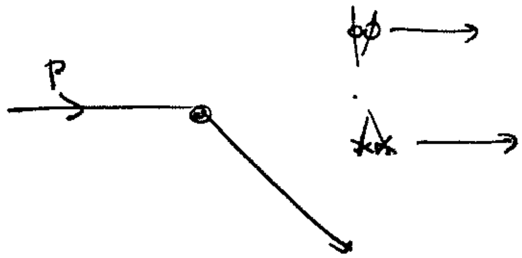
$$N_\gamma(\parallel \text{ to } \vec{\beta}') \cong \int dz\, f_\gamma(z)$$

where $z = \dfrac{kc}{E'}$ and $f_\gamma(z)$ is again the Weiszäcker-Williams distribution.

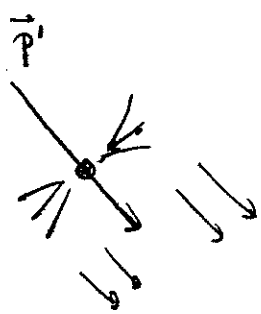
There is a very ~~interesting~~ physical interpretation of the Weiszäcker-Williams distribution. As a charged particle moves through free space, it carries its Coulomb field along with it. We saw earlier in the course ~~that~~ the Coulomb field is actually organized at times much earlier than the time the particle goes past.



We can think of the Coulomb field of an extremely relativistic particle as a collection of equivalent photons. When the charge is knocked out, the photons continue to go forward:



so we get a burst of photons moving parallel to $\vec{p}$.
When the Coulomb field forms around the outgoing particle, it leaves behind the same distribution of anti-photons, the final-state radiation.



There is an interesting test of this equivalent photon picture. In elementary particle physics, there are reactions that are naturally mediated by two photons. For example, since the π^0 meson decays to 2 photons, we have the reaction:

$$\gamma + \gamma \to \pi^0 \to \gamma + \gamma .$$

Now look for single π^0 production in e^+e^- collisions. One finds that the reaction

$$e^+e^- \to e^+e^- \; \pi^0$$
$$\hookrightarrow \gamma\gamma$$

Proceeds at the rate corresponding to the cross section

$$\sigma(e^+e^- \to e^+e^- \pi^0) \;=\; \int \frac{dx_1}{x_1}\, f_\gamma(x_1) \int \frac{dx_2}{x_2}\, f_\gamma(x_2)$$

$$\cdot\; \sigma\big(\gamma(E_1 = x_1 E) + \gamma(E_2 = x_2 E) \to \pi^0\big)$$

This is just the rate that is predicted by the picture in which the π^0 is formed from equivalent photons associated with the e^+ and the e^-.