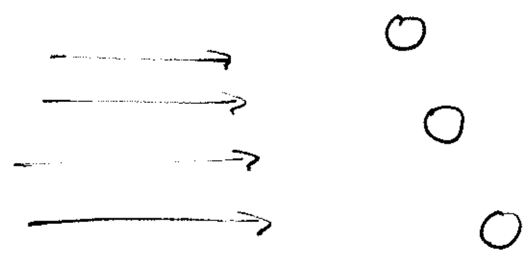
May 23

The Rainbow

In this lecture, I would like to describe another nontrivial result from the wave ~~theory~~ of light. This is the *rainbow*, one of the most remarkable optical phenomena.

The basic explanation for the rainbow goes back to the seventeenth century, to De Dominis (1611) and Descartes (1637). This theory is pure geometrical optics, but nevertheless it is very interesting. Certain additional aspects, however, require a wave picture and were not understood until the nineteenth century.

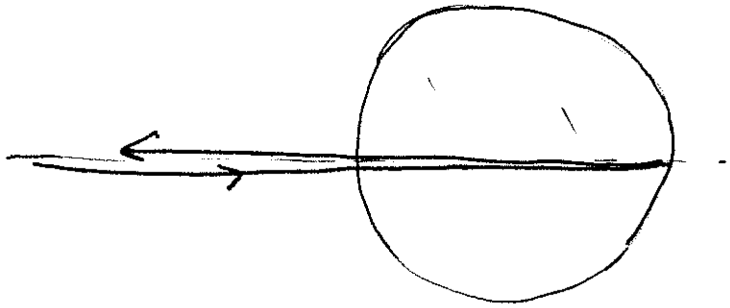
The rainbow is, first of all, an effect of sunlight reflected from water droplets in the atmosphere.



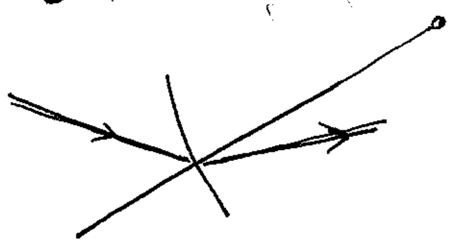
We can consider the water drops to be spherical. So how does a spherical drop lead to reflection at a fixed angle?

Let's first think about monochromatic light. Take the direction of incidence to be parallel to $\hat{z}$. We do not

know where on the surface of the sphere a given light ray will strike; actually, we should consider all possibilities.
The ray that hits at the equator reflects directly backward



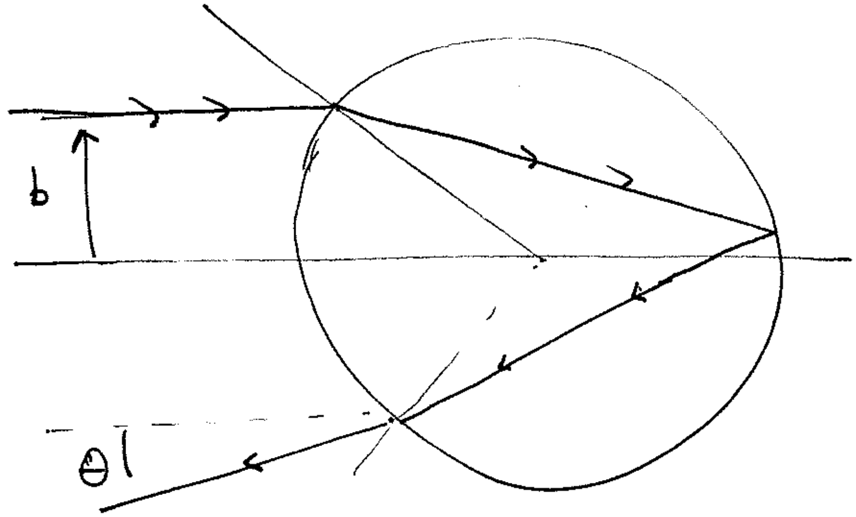
Let's follow a ray somewhat displaced from the equator. We must first take account of refraction at the surface of the drop. For visible light, the index of refraction of water is $n \sim 4/3$.



It will be relevant in a moment that the index of refraction increases somewhat with ω — since, as in Rayleigh scattering, higher ω brings us closer to the atomic resonances. Specifically

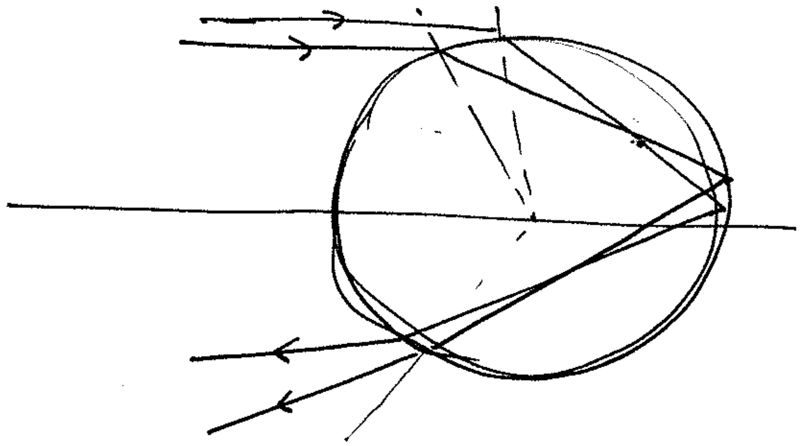
red	$\lambda = 700$ nm	$n =$	1.3309
green	500 nm		1.3364
blue	400 nm		1.3440

Now let's chase a ray that enters above the equator of the drop:

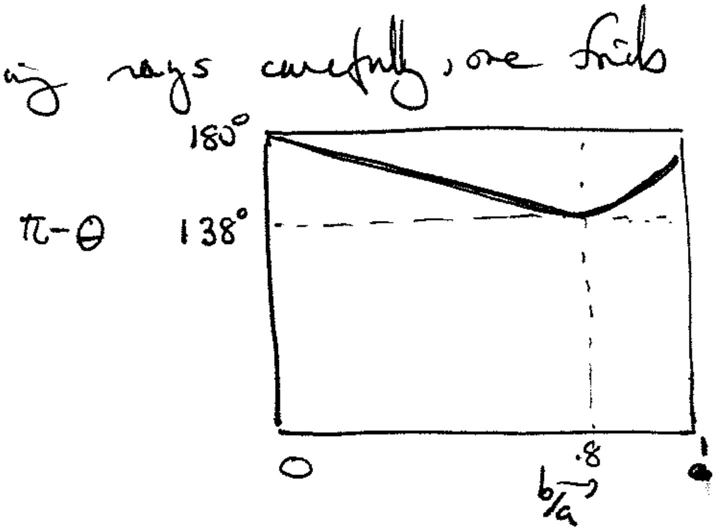


size of drop = a

As the impact parameter b increase from 0 to a, the reflection angle θ first increases, then decreases:



Tracing rays carefully, one finds



the maximum value of θ is about 42°

near the maximum ('Descartes' ray')

$$\theta = \theta_r - \alpha (b - b_r)^2$$

then if the values of b are distributed randomly over the sphere

$$\text{Prob.} = \int db\, b\, 2\pi \frac{1}{\pi a^2} = \int db\, b \frac{2}{a^2}$$

the probability distribution in θ is given by

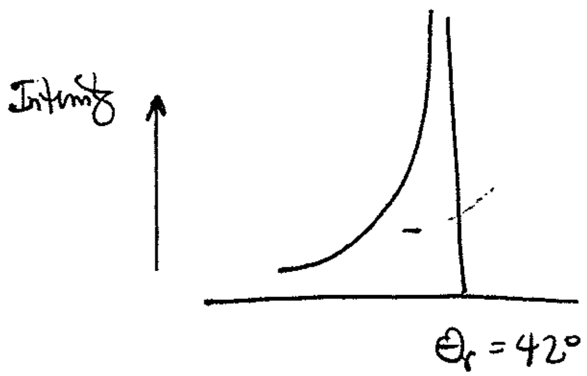
$$d\theta = -2\alpha (b - b_r)\, db$$

so

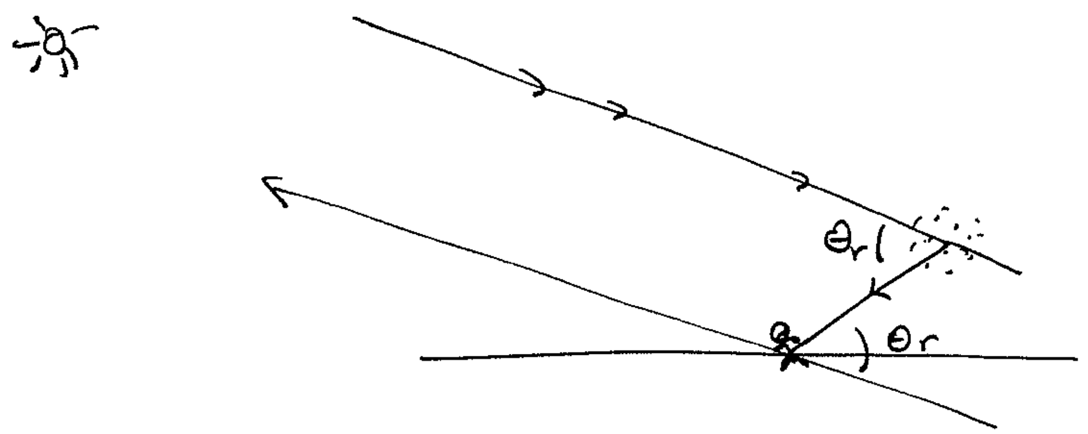
$$db\, b \sim \frac{d\theta}{[\theta_r - \theta]^{1/2}} \cdot b_r$$

and the ~~intensity~~ distribution is.

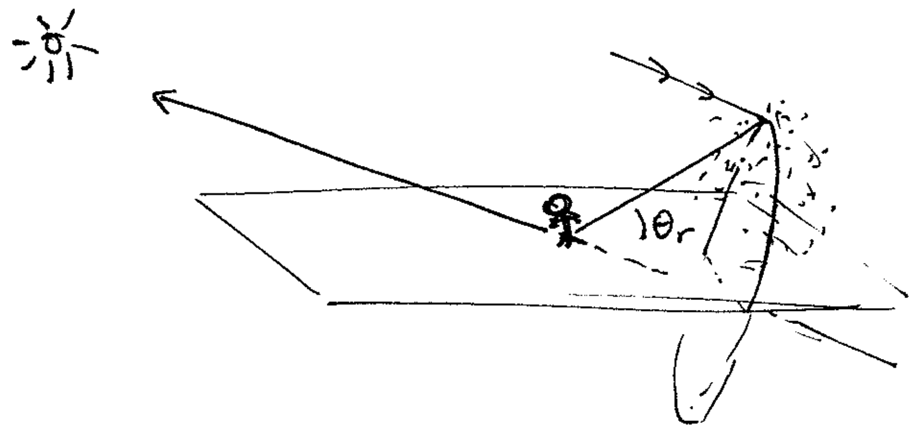
$$I(\theta) \sim \frac{d\theta}{[\theta_r - \theta]^{1/2}}$$



The singularity in this function appears on a cone of internal angle 42° around the direction opposite to the sun



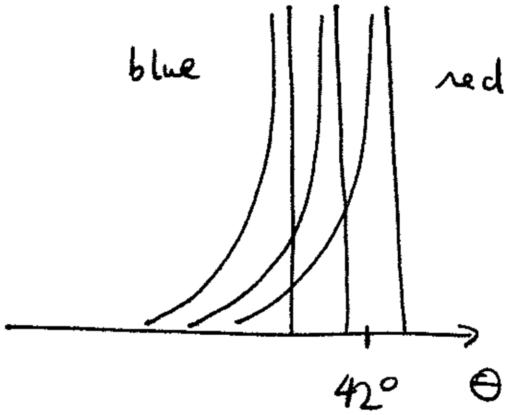
again

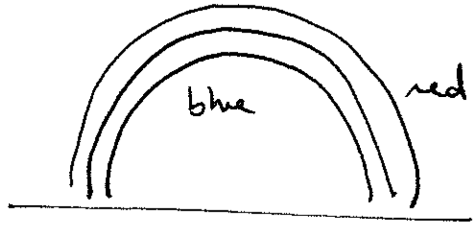


so it lights up a circular arc in the sky. As the index of refraction n increases, the angle θ_r decreases:

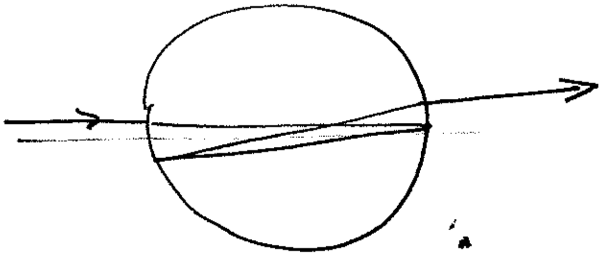
700 nm	$n = 1.3309$	42.38°	for comparison, the
500 nm	1.3364	41.27°	solar diameter
400 nm	1.3440	40.51°	is 0.5°

So what we see is a bow with separated colors, with blue on the inside and red on the outside:

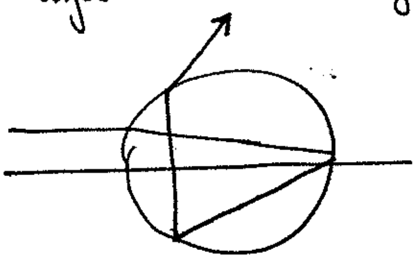




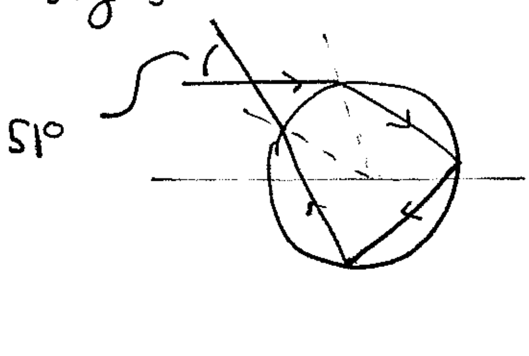
We will come back to the details of this diagram in a moment. First, though, consider the consequences of the light ray having another internal reflection:



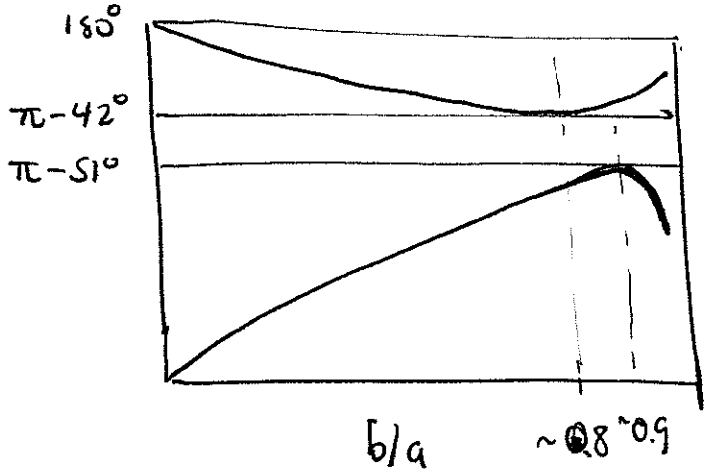
Here the ray with $b=0$ goes straight forward. As b increases the deflection angle becomes larger,



eventually going to a maximum at



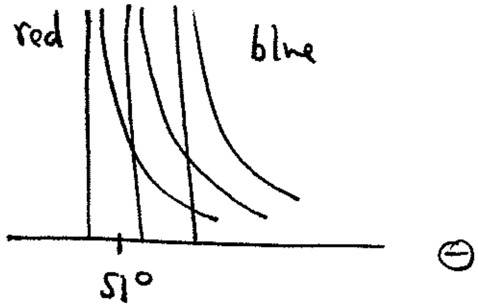
so, fitting in the previous diagram

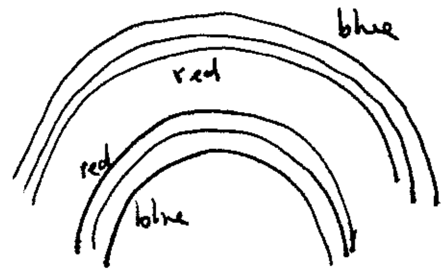


This leads to another cone of light, with a distribution of intensities singular at $\theta_r' \approx 51°$. The angle $(\pi - \theta_r')$ decreases as the index of refraction increases, so

700 nm	n = 1.3309	50.34°
500 nm	1.3364	52.33°
400 nm	1.3440	53.73°

This gives a secondary rainbow, with red inside and blue outside



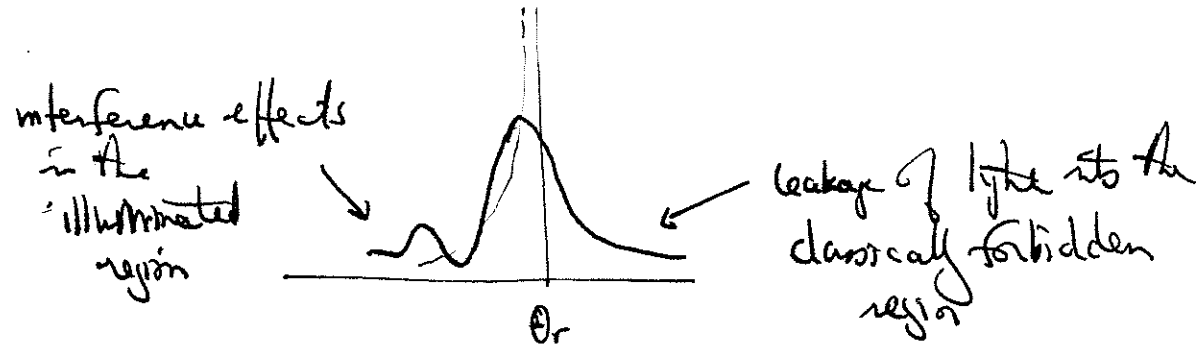


There is no ray with either one or two internal reflections that goes into the band with

$$42° < \Theta < 51°$$

Actually, when the double rainbow can be seen, this band is darker than the sky below the primary rainbow. The region is called Alexander's dark band (after Alexander of Aphrodisias (200 B.C.)).

Now let's come back to the apparent singular behavior of the intensity for each color of light at the corresponding θ_r. This is the result that comes from treating each ray as a particle trajectory. If instead we treat light as a wave, we will obtain modification of this picture



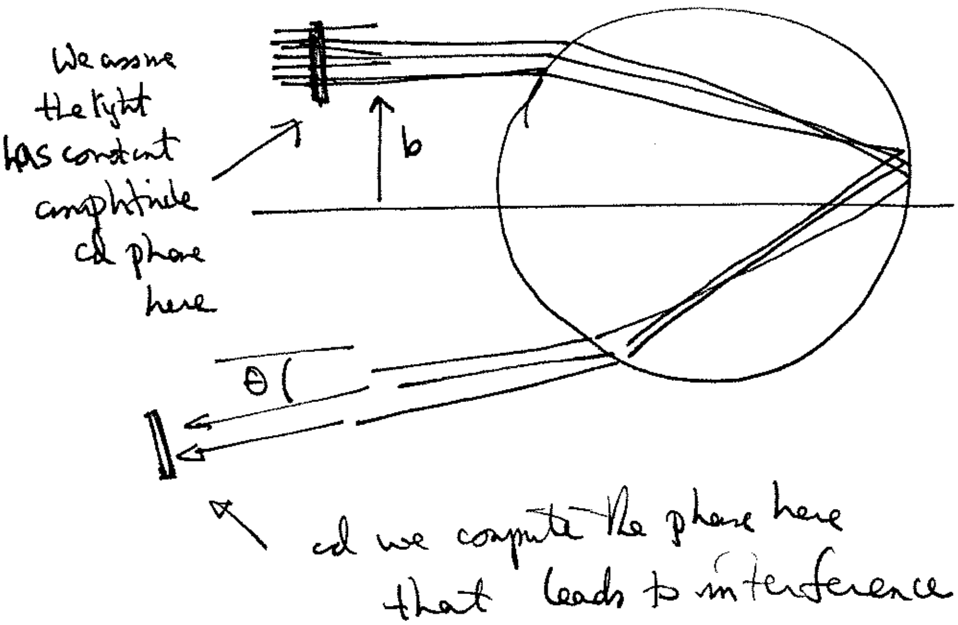
A ~~relatively~~ simple picture for treating these effects was introduced by George Airy (1839). What Airy tried to do was to account for the interference of nearby light rays going through the water droplet. On p.4, we wrote, essentially

$$I(\theta) \sim \left| \int db\, \pi b_r\, \delta(\theta - [\theta_r - \alpha (b_r - b)^2]) \right|^2$$

This is the ray-optics limit. However, in a theory where light is a wave, this expression must be generalized to something like

$$I(\theta) \sim \left| \int db \cdot (\text{const}) \cdot e^{i\,\Phi(b,\theta)} \right|^2$$

where Φ is the phase accumulated by the wave on a trajectory that enters the droplet with impact parameter b and exits in the direction θ:



In principle, one could compute the phase Φ for each trajectory in all detail, by considering the exact path each ray takes and the phases accumulated in reflection and refraction. This was done in detail by Mie (1908). However, there is a simpler consideration that gives approximately the right phase function. First of all

$$\Phi(b,\theta) = k \cdot \Delta l(b,\theta)$$

where Δl is the path length difference for various trajectories.

Now, the trajectories actually taken by light rays are those for which Δl is stationary:

$$\frac{\partial}{\partial b} \Delta l(b,\theta) = 0$$

This is just the Fermat principle of least time. So, I propose the form

$$\Delta l(b,\theta) = c\left[(\theta-\theta_r)\,\mathscr{b} + \tfrac{1}{3}\alpha\,\mathscr{b}^3\right]$$

where $\mathscr{b} = (b_r - b)$. This is the leading term in the expansion of Δl or Φ about $\theta=\theta_r$, $b=b_r$. We can check that it is correct by noting that

$$\frac{\partial}{\partial b}\Delta l = -c\left[\theta-\theta_r + \alpha(b_r-b)^2\right]$$

so that setting $\Delta l = 0$ gives

$$\theta = \theta_r - \alpha(b_r-b)^2$$

the correct ray-optics relation. Then (according to Airy):

$$I(\theta) \sim \left|\int_{-\infty}^{\infty} d\mathscr{b}\; e^{ikc\left[(\theta-\theta_r)\mathscr{b} + \frac{1}{3}\alpha\mathscr{b}^3\right]}\right|^2$$

We need to study this ~~integral~~ in the limit of short wavelength $\lambda \to 0$ or $k = \frac{2\pi}{\lambda} \to \infty$.

The ~~integral~~ cannot be done in closed form. So it is useful to define a standard form of this integral as a special function. Thus, let

$$Ai(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} du \quad e^{iuz + i\frac{u^3}{3}}$$

This is the Airy function. If we set $u = (kc\alpha)^{1/3} \theta$ in the integral on the previous page, we find

$$I(\theta) \sim \left| (\text{const}) \int du \quad e^{i\frac{(kc)^{2/3}(\theta-\theta_r)}{\alpha^{1/3}} u + \frac{i}{3}u^3} \right|^2$$

so

$$I(\theta) \sim \left| Ai\left(\frac{(kc)^{2/3}}{\alpha^{1/3}}(\theta - \theta_r)\right)\right|^2$$

To find the form of $I(\theta)$, we need to study the asymptotic behavior of $Ai(z)$ in both possible limits, $z \to -\infty$ & $z \to +\infty$. Let's start with $z \to -\infty$.

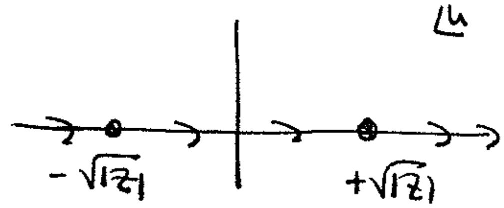
$Ai(z)$ is an integral over a pure phase $e^{i\phi(u)}$. So the dominant contribution to the integral comes from the regions of u that add up coherently, the regions where $\partial\phi/\partial u = 0$. This condition is:

$$\frac{\partial}{\partial u}\left(uz + \frac{u^3}{3}\right) = 0 \qquad \text{"stationary phase"}$$

$$z + u^2 = 0$$

For z negative:

$$u = \pm\sqrt{|z|}$$



Let's expand the exponent about the point $u = +\sqrt{|z|}$:

$$u = \sqrt{|z|} + v$$

$$uz + \tfrac{1}{3}u^3 = -|z|^{3/2} - \cancel{v|z|} + \tfrac{1}{3}|z|^{3/2} + \cancel{|z|v} + |z|^{\frac{1}{2}} v + \tfrac{1}{3}v^3$$

The terms linear in v cancel; this is just the stationary phase condition.

$$= -\tfrac{2}{3}|z|^{3/2} + |z|^{\frac{1}{2}} v^2 + \tfrac{1}{3} v^3$$

The integrand is then:

$$e^{i(uz + u^3/3)} = e^{-\frac{2}{3}|z|^{3/2}} \, e^{i|z|^{\frac{1}{2}}v^2 + \frac{i}{3}v^3}$$

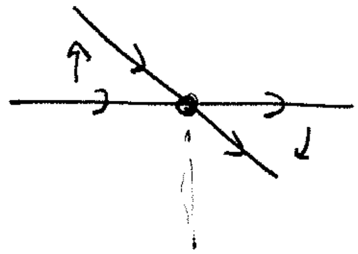
The integral over the term in v^2 is dominated by $|v| < \frac{1}{|z|^{1/4}}$, so the term in v^3 is systematically small, $\sim \frac{1}{|z|^{3/4}}$, as $z \to -\infty$.

A better way to understand the integral over v is to make the substitution

$$v = \frac{e^{+i\pi/4}}{|z|^{1/4}}\,\omega$$

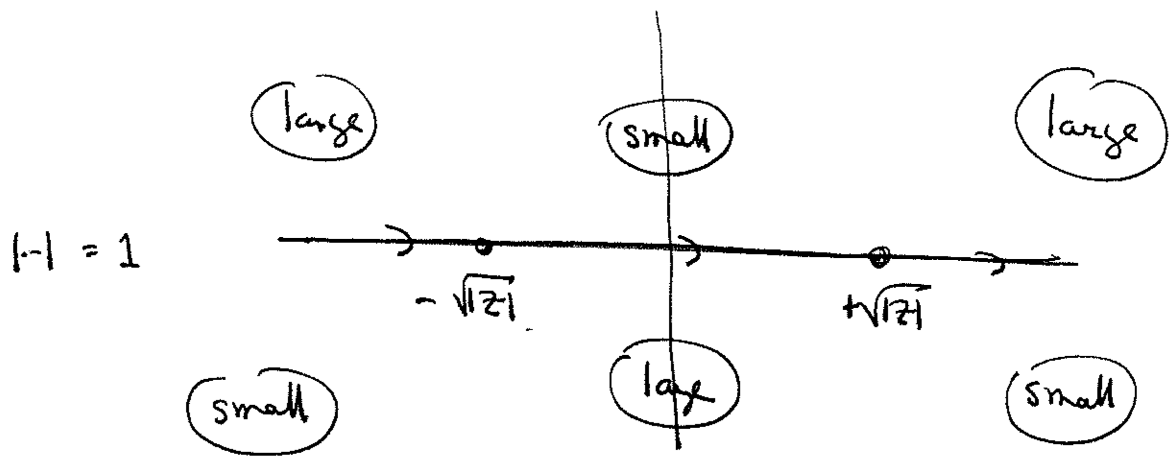
then
$$e^{i|z|^{\frac{1}{2}}v^2} = e^{-\omega^2}$$

If we integrate over ω, we ~~are~~ rotate the contour of ~~integration~~ in the complex u or v plane!

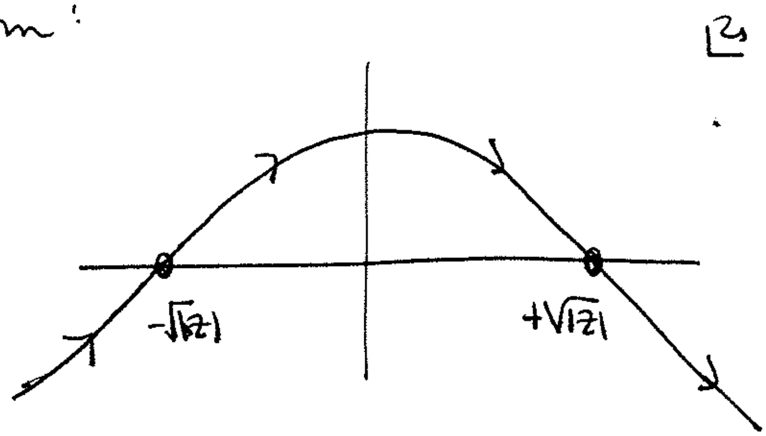


$u = +\sqrt{|z|}$

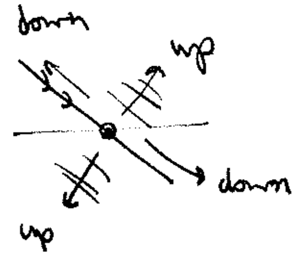
This is justified by the fact that $\left|e^{i(uz + z^3/3)}\right|$ varies in the complex plane:



Since the value of the ~~integral~~ over u is independent of what contour we use to evaluate it, it makes sense to choose a contour on which the integrand is as small as possible. Then there are no large cancellations between different regions. In this case, we want to push the contour:



The vicinity of $u = \sqrt{|z|}$ is a maximum along the new contour, and a minimum in the direction orthogonal to the contour — a saddle point:



The contribution to $Ai(z)$ from the neighborhood of $u = +\sqrt{|z|}$ is then

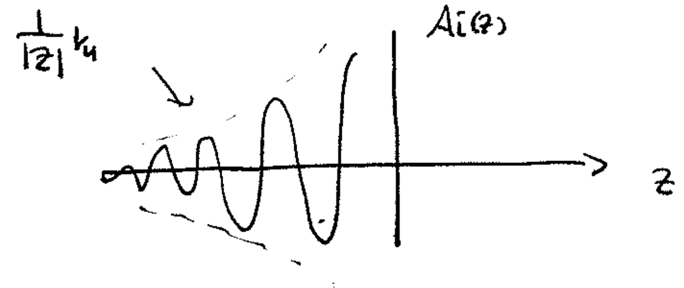
$$\frac{e^{-i\frac{2}{3}|z|^{3/2}}}{2\pi} \; \frac{e^{+i\pi/4}}{|z|^{1/4}} \int dw \; e^{-w^2}$$

$$= \frac{1}{2} \, e^{-i\left(\frac{2}{3}|z|^{3/2} - \frac{\pi}{4}\right)} \frac{1}{\sqrt{\pi}} \frac{1}{|z|^{1/4}}$$

The contribution from the stationary phase point at $u = -\sqrt{|z|}$ is exactly the complex conjugate of this — as you can

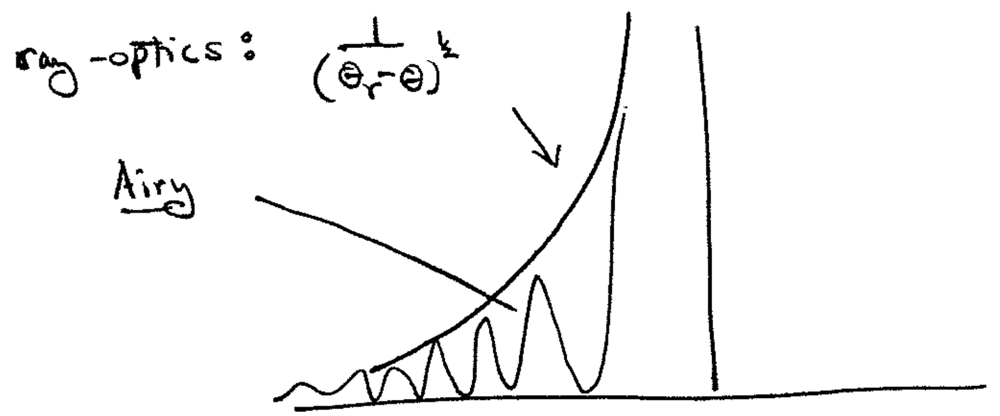
Show then

$$Ai(z) \cong \frac{1}{\sqrt{\pi}} \frac{1}{|z|^{1/4}} \cos\left(\frac{2}{3}|z|^{3/2} - \frac{\pi}{4} \right)$$



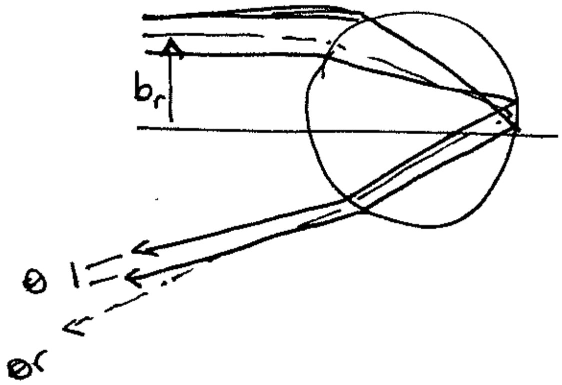
Putting this back into the expression on p. 11,

$$I(\theta) \sim \frac{1}{(\theta_r - \theta)^{1/2}} \cos^2\left(\frac{2}{3} \frac{kc}{\sqrt{\alpha}} (\theta_r - \theta_r)^{3/2} - \frac{\pi}{4} \right)$$



It is interesting to ask what is going on physically. The ~~stationary~~ phase points are solutions to the classical ray-optics equations. So $u = \pm\sqrt{|z|}$ correspond to the two trajectories that give $\Theta = \theta < \theta_r$, one with $b < b_r$, one with $b > b_r$. It is the interference between these two rays that gives

the pattern observed ~ the classically illuminated region
$\theta < \theta_r$

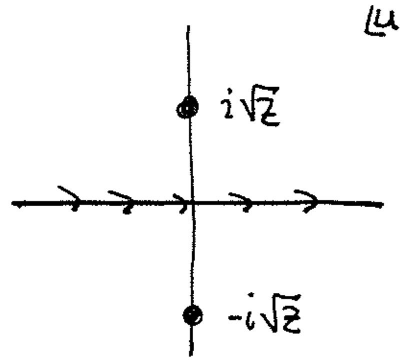


Now let's study the Airy function as $z \to \infty$. This corresponds to the classically forbidden region where no rays go. So, there should be no solutions to the stationary phase conditions. More correctly, there are no solutions for real u.

$$\frac{\partial}{\partial u}\left(uz + \frac{u^3}{3} \right) = 0$$

$$\Downarrow$$

$$z + u^2 = 0$$



for $z > 0$ $\quad u = \pm i\sqrt{z}$

Let's expand the exponent of the integrand in the vicinity of $u = +i\sqrt{z}$

$$u = +i\sqrt{z} + v$$

$$uz + \frac{u^3}{3} = i(z)^{3/2} + \cancel{zv} - i\frac{z^{3/2}}{3} - \cancel{zv} + i\sqrt{z}\,v^2 + \frac{v^3}{3}$$

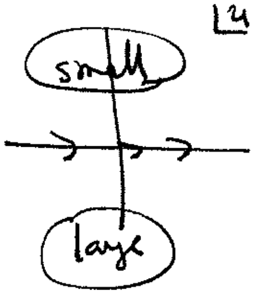
$$= i\tfrac{2}{3}|z|^{3/2} + i\sqrt{z}\,v^2 + v^3/3$$

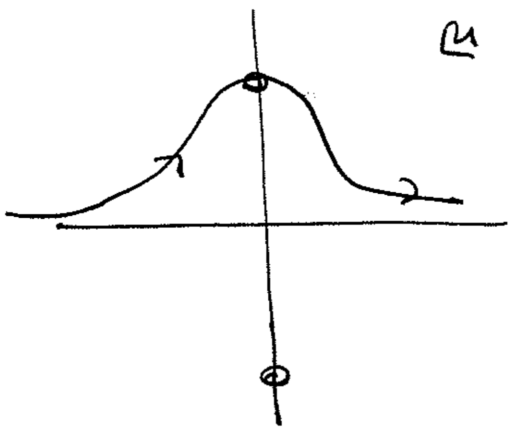
again $v^3 \sim \frac{1}{|z|^{3/4}}$ and can be ignored in the first approximation.

Then

$$e^{i(uz + \frac{u^3}{3})} \cong e^{-\frac{2}{3}z^{3/2}} \cdot e^{-\sqrt{z}\,v^2}$$

so this function is small near $u = +i\sqrt{z}$, with the crest of a hill at $u = +i\sqrt{z}$, $v = 0$. Conversely, near $u = -i\sqrt{z}$, the integrand is large $\sim e^{+\frac{2}{3}z^{3/2}}$. Since the result of the integral is independent of the contour, it pays to deform the contour to pass through $u = +i\sqrt{z}$



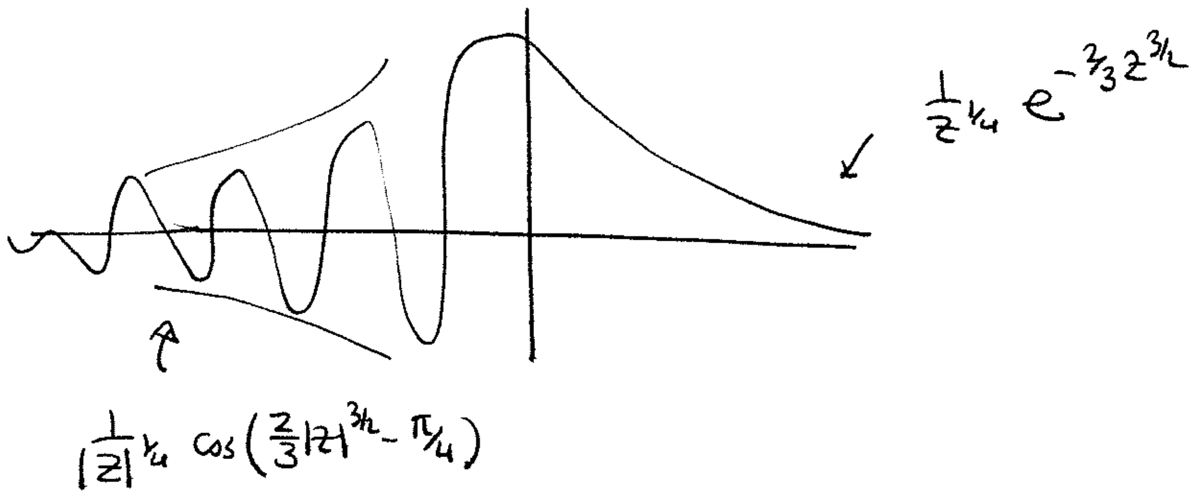


then

$$Ai(z) \cong \frac{1}{2\pi}\, e^{-\frac{2}{3}z^{3/2}} \int_{-\infty}^{\infty} dv\; e^{-\sqrt{z}\,v^2}$$

$$\cong \frac{1}{2}\frac{1}{\sqrt{\pi}}\frac{1}{z^{1/4}}\, e^{-\frac{2}{3}z^{3/2}} \qquad \text{as } z \to \infty$$

now we have a global picture of $Ai(z)$



The Airy function $Ai(z)$ comes up in many other applications in which waves cross ~~into~~ a classically forbidden region. For example, you will meet it again in quantum mechanics in the discussion of tunnelling through a classical barrier. So it might be useful to know a few more things about it:

① $Ai(z)$ solves a simple differential equation:

$$\left(\frac{d^2}{dz^2} - z \right) Ai(z) = 0$$

② $Ai(z)$ is really a Bessel function in disguise:

for $-z = y > 0$

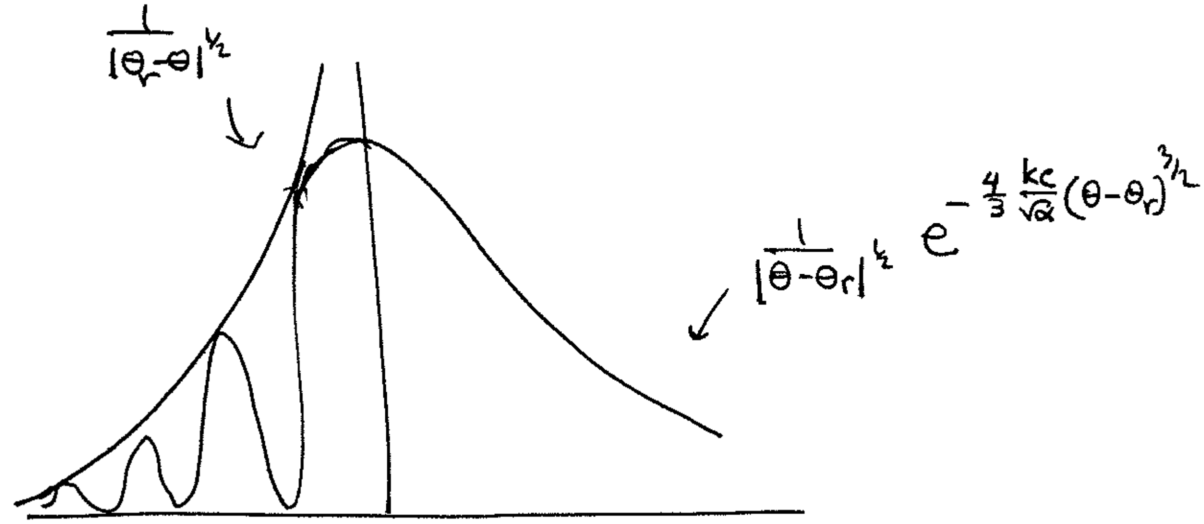
$$Ai(z) = \frac{y^{1/2}}{3^{4/3}} \left\{ J_{1/3}\left(\tfrac{2}{3} y^{3/2}\right) + J_{-1/3}\left(\tfrac{2}{3} y^{3/2}\right) \right\}$$

for $z > 0$

$$Ai(z) = \frac{\sqrt{z}}{\pi\, 3^{5/6}} K_{1/3}\left(\tfrac{2}{3} z^{3/2}\right)$$

The final picture of $I(\theta) \propto \left| Ai\left(\frac{(kc)^{2/3}}{\alpha^{1/3}} (\theta - \theta_r) \right) \right|^2$

is



More accurate theories of the rainbow have been developed. It turns out ~~that~~ a complete treatment of the interference of trajectories using Mie's theory fills in the zeros of the Airy function to some extent. An elegant method of summing over Mie's expressions was introduced very recently (1969) by Nussenzveig, using the method of "complex angular momentum" invented for applications in elementary particle theory. Thus the theory of the rainbow spans the whole history of physics!

I would like to recommend two books with beautiful pictures of atmospheric optical phenomena and descriptions of their physical origin:

R. Greenler, Rainbows, Halos, and Glories

DK Lynch and W. Livingston, Color and Light in Nature