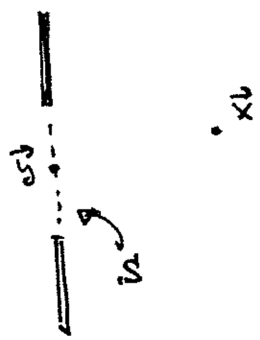
May 18

Holography

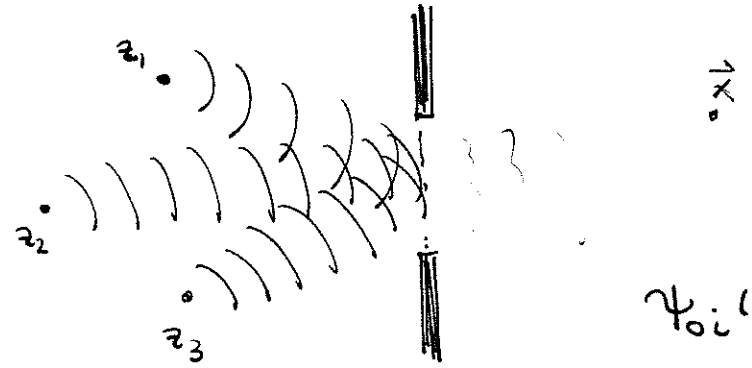
Once you understand the theory of diffraction, it is straightforward to understand how holograms work. I'll describe a little of the theory of holography in this lecture.

We continue to concentrate on the case of scalar waves of definite frequency ω, $k = \omega/c$. For this situation, if we have a screen S and a region to its right, the Kirchhoff integral tells us ~~that~~

$$\psi_0(\vec{x}) = \int_S d^2y\, \hat{n} \left\{ \frac{e^{ik|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|} \vec{\nabla}_y \psi_0(y) - \vec{\nabla}_y \left(\frac{e^{ik|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|} \right) \psi_0(\vec{y}) \right\}$$



so that the radiation field in the whole region to the right of the screen S is determined by the boundary values on S. In particular, if a number of objects are placed to the left of the screen and radiate onto the screen:



$$\psi_{0i}(\vec{y}) = \psi_{0i} \frac{e^{ik|\vec{y}-\vec{z}_i|}}{4\pi|\vec{y}-\vec{z}_i|}$$

Then in the approximation appropriate to the Fresnel region,

$|\vec{z}_i| \gg a$, where a is the size of the aperture, and $a \gg \lambda$,

and forward angles,

$$\psi_0(\vec{x}) \cong \frac{1}{2\pi} \int_S d^2y \left(\sum_i \psi_{0i} \frac{e^{ik|\vec{y}-\vec{z}_i|}}{|\vec{z}_i|} \right) \frac{e^{ik|\vec{x}-\vec{y}|}}{|\vec{x}|}$$

This equation gives rise to a very ~~interest~~ idea, due to Dennis Gabor (1948): In an ordinary photography, the paper records intensities but it does not keep the information about the phase of the light wave. But what if we could faithfully capture the phase of the wave on S? Then, as we move around in the region to the right of S and look at the screen, we would see the radiation field produced by the three-dimensional array of sources. This radiation would contain the full three-dimensional information about the sources' positions. As we moved, the sources would change their relative positions as perspective requires. If we look with two eyes, we will have the usual impression of depth that we get from binocular vision. So, if we can record both the magnitude and phase of $\psi_0(\vec{y})$ — just on the screen S — we will have a fully 3-~~dimensional~~ photograph!

What if we could implant on the screen, not the full complex function

$$\sum_i \psi_{0i} \frac{e^{ik|\vec{y}-\vec{z}_i|}}{|z_i|}$$

but only the real part of this function? Then the wave at $\vec{x}$ would be that resulting from putting

$$\frac{1}{2}\sum_i \psi_{0i} \frac{e^{ik|y-z_i|}}{|z_i|} + \frac{1}{2}\sum_i \psi_{0i}^* \frac{e^{-ik|y-z_i|}}{|z_i|}$$

into the Kirchhoff integral. The first term gives the radiation field created by $z_1, z_2, \cdots$. To find the interpretation of the second term, expand for large z_i^z:

$$|y-z_i| \cong \left[(z_i^z)^2 + ((y-z_i)^x)^2 + ((y-z_i)^y)^2\right]^{\frac{1}{2}}$$

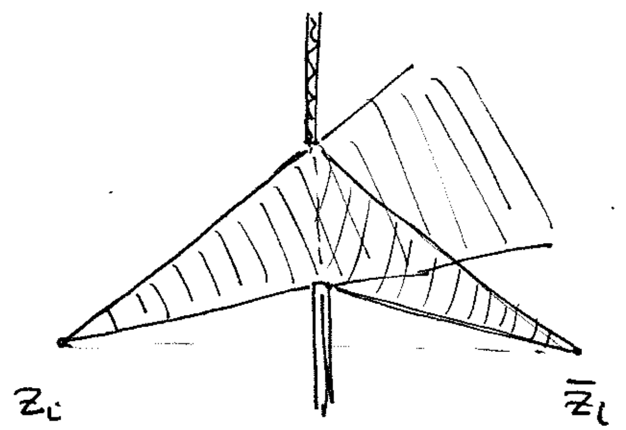
$$= |z_i^z| + \frac{1}{2}\frac{[((y-z_i)^x)^2 + ((y-z_i)^y)^2]}{|z_i^z|} - \frac{1}{8}\frac{[((y-z_i)^x)^2+((y-z_i)^y)^2]^2}{(z_i^z)^3}$$

$$+ \cdots$$

so changing the sign of i in the exponent is equivalent to sending $z_i^z \to -z_i^z$. This rough argument indicates that the corresponding term in the Kirchhoff integral gives a wave that converges to a focus at $\bar{z}_i = (z_i^x, z_i^y, -z_i^z)$, which is a point to the right of the screen. Thus, for each source to the left of the screen S, the pattern

$$Re\left(\psi_{0i} \frac{e^{ik|y-z_i|}}{|z_i|} \right)$$

in the Kirchhoff integral gives



For an observer to the right of the screen, there is a <u>virtual</u> <u>image</u> at z_i and a <u>real</u> <u>image</u> at $\bar{z}_i$. An observer to the right of the screen but to the left of $\bar{z}_i$, looking at the screen, will see only the virtual image.

Now it is clear how to procede. Arrange ~~that~~ objects at $z_1, z_2, \cdots$ reflect light from a coherent light source. And, shine on the screen in addition a strong source of light that is also coherent with the light that illuminates the objects. For simplicity, take this to be a plane wave:

$$\Psi_0 e^{ik\vec{y}\cdot\hat{z}} \qquad (\Psi_0 \text{ real})$$

so that the phase is a constant on S. Then the intensity on S is proportional to

$$\left| \Psi_0 + \sum_i \psi_{0i} \frac{e^{ik|y-z_i|}}{|z_i|} \right|^2$$

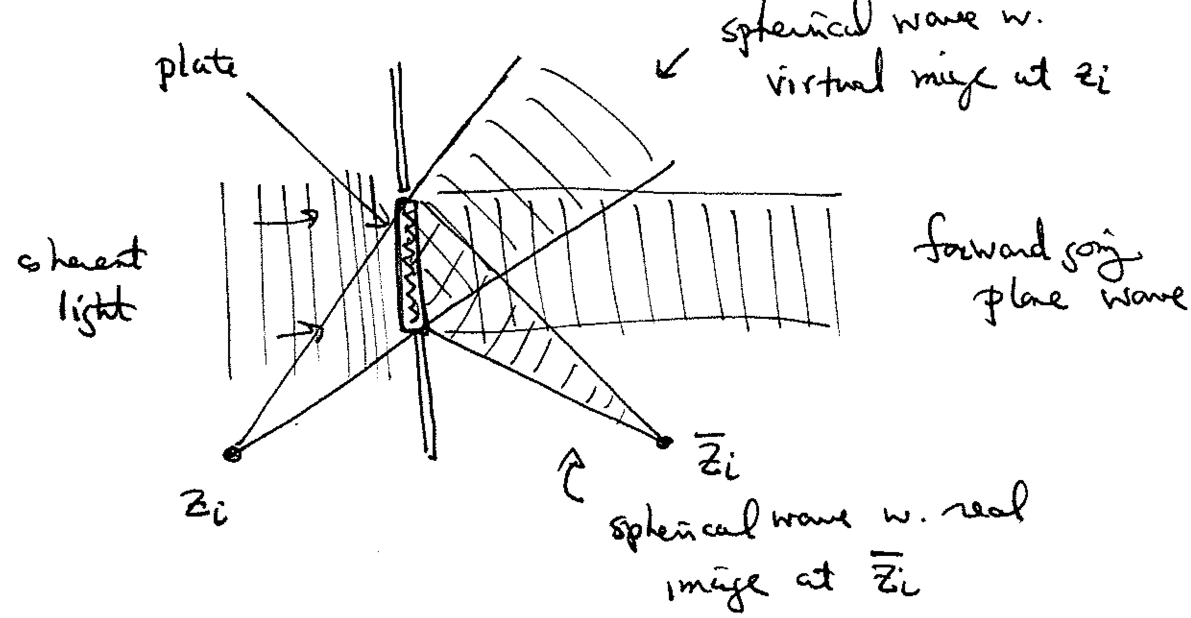
$$= |\Psi_0|^2 + 2\Psi_0 \, \mathrm{Re}\left(\sum_i \psi_{0i} \frac{e^{ik|y-z_i|}}{|z_i|} \right)$$

$$+ \quad O(\psi_{0i}^2)$$

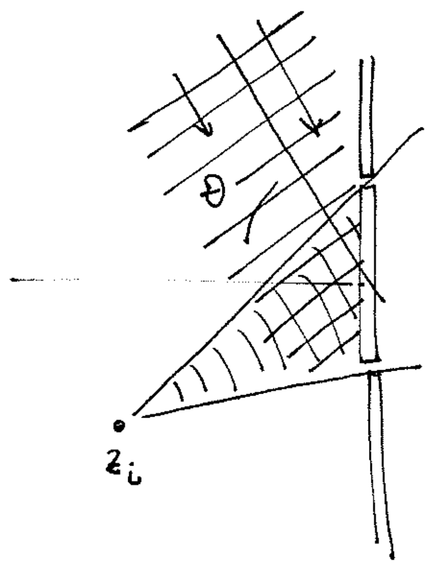
assume that the terms in ψ_{0i}^2 can be ignored. Now put a photographic plate along S and arrange that this plate is darkened proportional to the light intensity. Finally, take away all of the sources and pass coherent light through the developed plate. This gives a wave whose value just to the right of the screen is

$$\Phi_0 \left\{ 1 - \varkappa \left(|\Psi_0|^2 + 2\Psi_0 \, \mathrm{Re} \sum_i \psi_{0i} \frac{e^{ik|y-z_i|}}{|z_i|} \right) \right\}$$

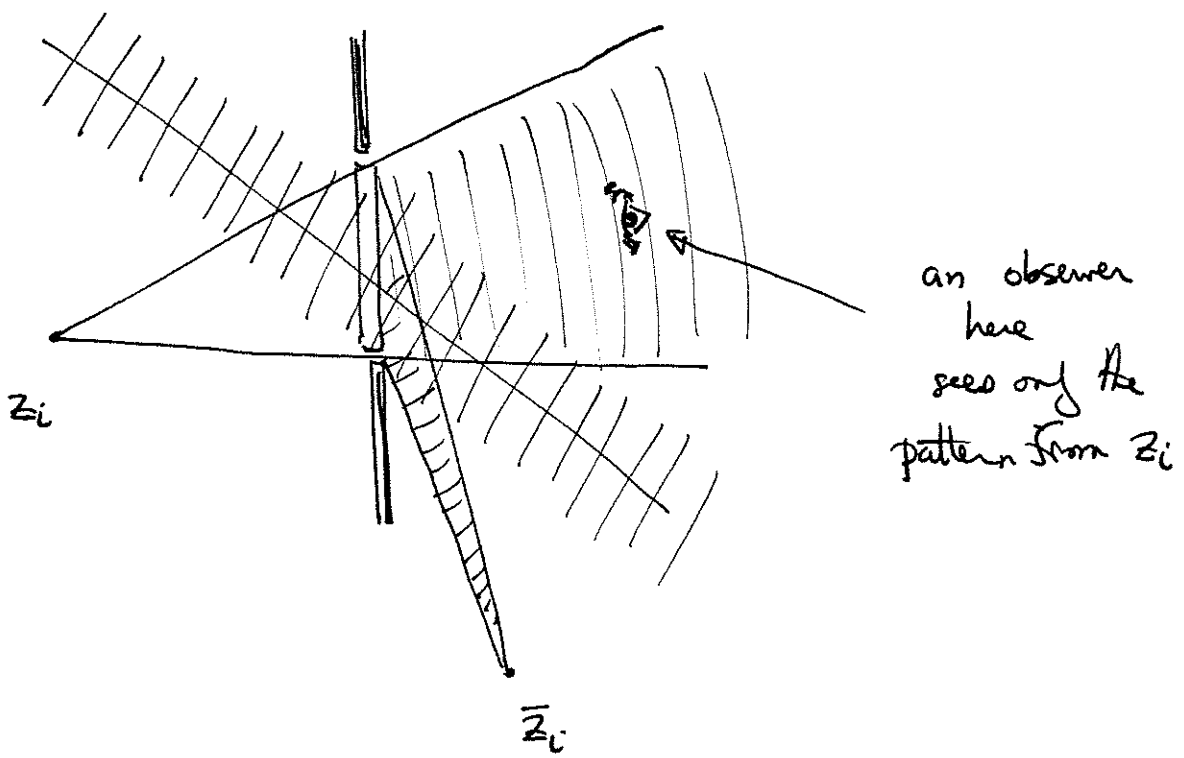
which we recognize as producing



If we could then ignore the forward-going plane wave and the real images, we have a 3-dimensional photograph of the sources z_i. This might be easier if we use the configuration:



which, when illuminated from the same direction, yields

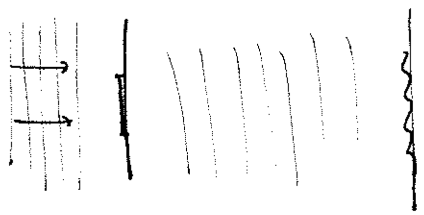


[To deduce the position of the real image, reflect the component of z_i parallel to $\vec{k}_0$, the wavevector of the illuminating plane wave.]

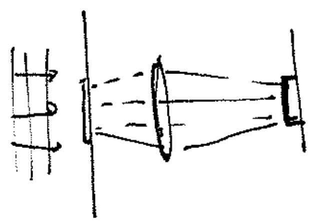
This geometry is due to Leith and Upatnieks, who played a key role in reviving Gabor's ideas in 1962. Gabor's ideas were well before their time; in particular, in 1948 there was no source of coherent light with long enough coherence lengths in space and time to allow holographic photography. Leith and Upatnieks realized that the newly invented *laser* provided a source of coherent light that would make holography practical.

Once this technology is available, many applications follow, some obvious, some less so. An "obvious" application that should be pointed out is the ability to perform interference experiments involving objects present at different times, as long as these times are within the coherence time of the laser. For example, one can shine a laser through air, then flash the laser again as a bullet passes, obtaining the interference fringes corresponding to the heating of the air by the bullet:

This picture comes from M. Françon's book <u>Holography</u>, which also includes a much more nontrivial application. Consider a character (考) drawn on a transparency. Passing light through the transparency, form the Fraunhofer diffraction pattern:



or



Add a plane wave of large amplitude, and make the holograph. Let the image of the character be $f(x,y)$ (i.e. $f = 1$ for points (x,y) where the image is black, and $f = 0$ for points (x,y)

where the image is white. Then the wave amplitude on the plane of the photographic plate is

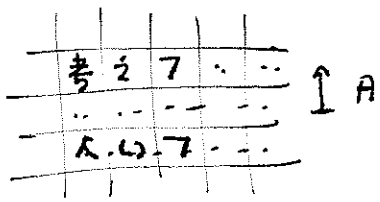
$$\tilde{f}(\alpha x, \alpha y)$$

where $\alpha = \frac{k}{z}$ with no lens, $\alpha = b\frac{k}{z}$ with the lens.

Then the intensity recorded on the plate is

$$\Phi_0 \left\{ 1 - \kappa\left(|\Phi_0|^2 + \Phi_0 [\tilde{f}^*(\alpha x, \alpha y) + \tilde{f}(\alpha x, \alpha y)] \right) \right\}$$

Now develop the plate and put up another transparency with a whole sheet of characters:



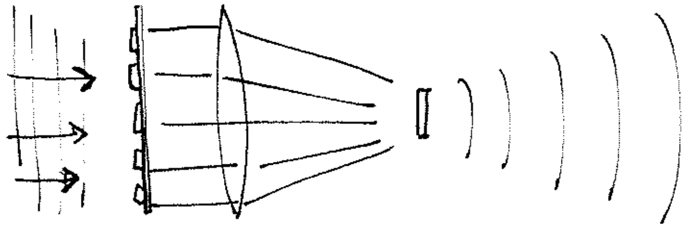
The ~~intensity~~ pattern of this sheet is

$$\sum_{ij} f_{ij}(x - iA, y - jA) \qquad f_{ij} = \text{character at } ij$$

Taking the Fraunhofer diffraction pattern of this transparency, we find the Fourier transform

$$\sum_{ij} \tilde{f}_{ij}(k_x, k_y)\, e^{-ik_x(iA) - ik_y(jA)}$$

Now, pass this wave pattern thrgh the hologram:



The wave ~~that~~ passes thrgh has the form

$$\sum_{ij} \tilde{f}_{ij}(\alpha x, \alpha y)\ e^{-i(\alpha x)(iA) - i(\alpha y)(jA)}$$

$$\cdot \left\{ (\text{const}) + \kappa \psi_0 \left[\tilde{f}(\alpha x, \alpha y) + \tilde{f}^*(\alpha x, \alpha y) \right] \right\}$$

This in particular contains the form

$$\tilde{f}_{ij}(\alpha x, \alpha y)\ f^*(\alpha x, \alpha y)\ e^{-i(\alpha x)(iA) - i(\alpha y)(jA)}$$

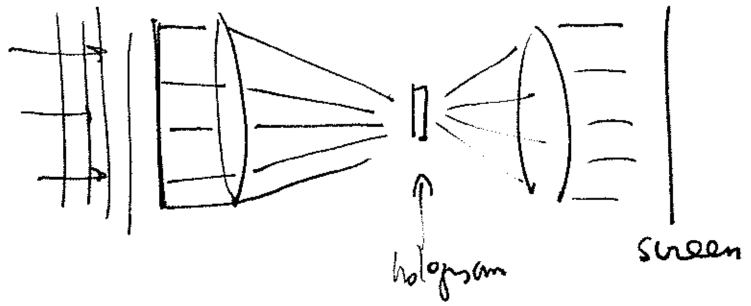
if the character at ij is the same as the original character

考

then we have $\underbrace{|f(\alpha x, \alpha y)|^2}\ e^{-i[(\alpha x)\cdot A\cdot i + (\alpha y)(jA)]}$

and approximating the prefactor by a constant, the Fourier transform of this is a bright spot at the position (i,j)

8. The setup



puts on the screen a set of spots at the positions occupied by the character ~~that~~ was to be recognized!
This is only one of many applications that exploits the Fourier relation of objects and their diffraction patterns or holograms.