May 16

Fresnel Diffraction

In the previous lecture, I remarked that the Fraunhofer approximation is good only sufficiently far from the aperture, at a distance

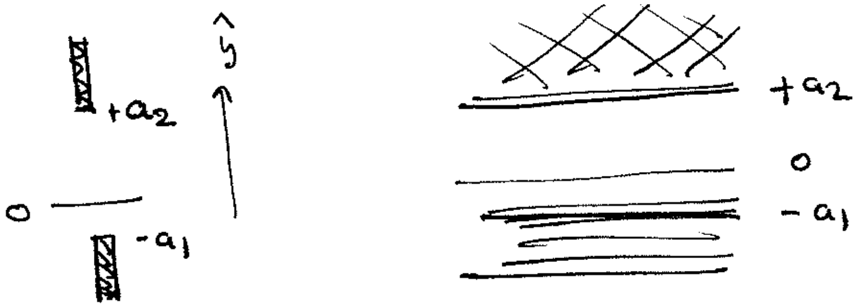
$$r \gg \frac{a^2}{\lambda}$$

To find the diffraction pattern closer in, we need to keep more terms in the series expansion of

$$k|\vec{x}-\vec{y}|$$

It is very instructive to compute just the next approximation, in which we keep terms quadratic in the source position y. This is the Fresnel approximation.

For simplicity, I'll just consider the simple linear slit



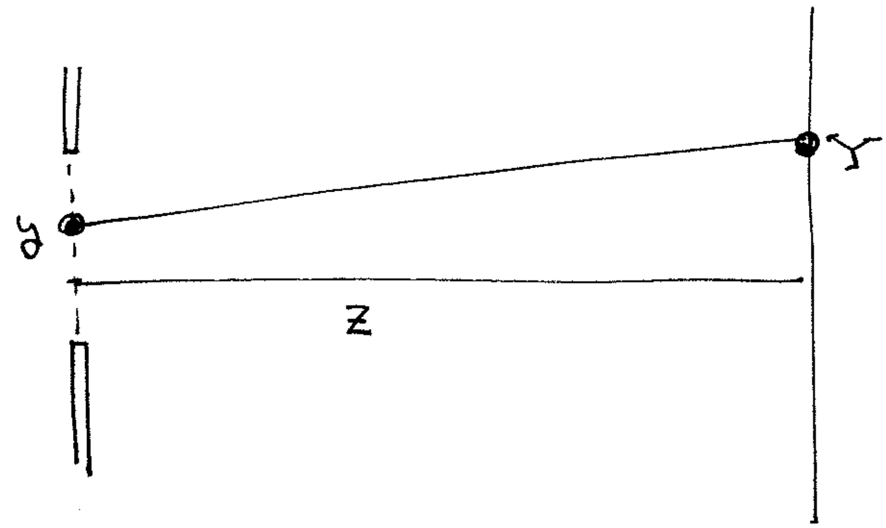
running from $-a_1$ to a_2. The integral from the Kirchhoff diffraction theory is

$$\int_{-a_1}^{a_2} d^2y \; e^{ik|\vec{x}-\vec{y}|}$$

Previously, we considered the point $\vec{x}$ to be very far away and approximated

$$e^{ik|x-y|} \cong e^{ikx}\, e^{-ik\hat{x}\cdot\vec{y}}$$

with further approximations for $\vec{x}$ in the forward direction. Let's set up the forward geometry more carefully:



If also $|Y| \ll Z$,

$$e^{ik|x-y|} \cong e^{ik\left(Z + \frac{1}{2}\frac{Y^2}{Z} + \dots\right)}\, e^{-ik\frac{Y}{Z}y}$$

I would like to improve this slightly by writing

$$|\vec{x}-\vec{y}| = \left[Z^2 + (Y-y)^2\right]^{\frac{1}{2}}$$

$$\cong Z + \frac{1}{2}\frac{(Y-y)^2}{Z}$$

or

$$e^{ik|\vec{x}-\vec{y}|} \cong e^{ikZ}\, e^{i\frac{k}{2}\frac{1}{Z}(Y-y)^2}$$

appropriate to the region $|y| \sim |Y| \ll Z$

but not necessarily $Z \gg \frac{|y|^2}{\lambda}$

The diffraction integral then becomes

$$e^{ikZ} \int_{-a_1}^{a_2} dy \; e^{i\frac{k}{2Z}(y-Y)^2}$$

let u be defined by $\frac{\pi}{2}u^2 = \frac{k}{2Z}(y-Y)^2$

Then (dropping the phase e^{ikZ}) the integral becomes

$$\sqrt{\frac{\pi Z}{k}} \int_{+u_1}^{u_2} du \; e^{i\frac{\pi}{2}u^2} = \sqrt{\frac{\lambda Z}{2}} \int_{+u_1}^{u_2} du \; e^{i\frac{\pi}{2}u^2}$$

where $u_1 = \sqrt{\frac{2}{\lambda Z}}(-a_1 - Y)$ $\qquad u_2 = \sqrt{\frac{2}{\lambda Z}}(a_2 - Y)$

The ~~integrals~~ cannot be done in closed form, so it is useful to define them as special functions:

$$\int_0^{u_0} du \; e^{i\frac{\pi}{2}u^2} = C(u_0) + i\,S(u_0)$$

i.e.

$$C(u_0) = \int_0^{u_0} du \cos\left(\frac{\pi}{2}u^2\right)$$

$$S(u_0) = \int_0^{u_0} du \sin\left(\frac{\pi}{2}u^2\right)$$

the Fresnel integrals

for fixed Z and varying Y, the intensity of the diffraction pattern is given by

$$I(Y) \propto |C(u_2) - C(u_1)|^2 + |S(u_2) - S(u_1)|^2$$

Let's see what we can learn about C and S and then put together a picture of the intensity pattern.

First of all,

at $u_0 = 0$ $\qquad C(0) = S(0) = 0$

as $u_0 \to \infty$ $\qquad \int_0^\infty du\, e^{i\frac{\pi}{2}u^2} = \int_0^\infty du\, e^{-(-i\frac{\pi}{2})u^2}$

$$\left(\begin{array}{l}\text{assuming } e^{i\frac{\pi}{2}u_0^2} \to 0 \\ \quad \text{as } u_0 \to \infty \\ \text{i.e. } u_0 \to \infty(1+i\epsilon)\end{array}\right) \quad = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{-i\frac{\pi}{2}}} = \frac{1}{\sqrt{2}}\cdot(i)^{1/2} = \frac{1+i}{2}$$

so $\qquad C(\infty) = \frac{1}{2} \qquad S(\infty) = \frac{1}{2}$

as $u_0 \to -\infty$ $\qquad \int_0^{-\infty} du\, e^{i\frac{\pi}{2}u^2} = -\int_0^\infty dw\, e^{i\frac{\pi}{2}w^2}$

$$= -\frac{1+i}{2}$$

in fact $\qquad C(-u) = -C(u) \qquad S(-u) = -S(u)$

C and S are <u>odd</u> functions

Near $u_0 = 0$

$$\int_0^{u_0} du \; e^{i\frac{\pi}{2}u^2} \cong \int_0^{u_0} du \left(1 + i\frac{\pi}{2}u^2 + \cdots\right)$$

$$\cong u_0 + i\frac{\pi}{6}u_0^3 + \cdots$$

so

$$C(u_0) \approx u_0 \qquad S(u_0) \approx \frac{\pi}{6}u_0^3 \quad \text{as } u_0 \to 0$$

Near $u_0 = \infty$

$$\int_0^{u_0} du \; e^{i\frac{\pi}{2}u^2} = \int_0^{\infty} du \; e^{i\frac{\pi}{2}u^2} - \int_{u_0}^{\infty} du \; e^{i\frac{\pi}{2}u^2}$$

the second ~~integral~~ can be expanded by integration by parts:

$$\int_{u_0}^{\infty} du \; e^{i\frac{\pi}{2}u^2} = \int_{u_0}^{\infty} du \; \frac{1}{i\pi u}\left(i\pi u \, e^{i\frac{\pi}{2}u^2}\right)$$

$$= \int_{u_0}^{\infty} du \; \frac{1}{i\pi u} \frac{d}{du}\left(e^{i\frac{\pi}{2}u^2}\right)$$

$$= \frac{1}{i\pi u} e^{i\frac{\pi}{2}u^2}\Big|_{u_0}^{\infty} - \int_{u_0}^{\infty} du \; \frac{-1}{i\pi u^2} e^{i\frac{\pi}{2}u^2}$$

using $e^{i\frac{\pi}{2}u_0^2} \to 0$ as $u_0 \to \infty(1+i\epsilon)$

$$= -\frac{1}{i\pi u_0} e^{i\frac{\pi}{2}u_0^2} - \frac{1}{(i\pi)^2}\frac{2}{u_0^3} e^{i\frac{\pi}{2}u_0^2} + \cdots$$

keeping just the first term

$$\int_0^{u_0} du\, e^{i\frac{\pi}{2}u^2} \cong \frac{1+i}{2} - \frac{i}{\pi u_0}\, e^{i\frac{\pi}{2}u_0^2} + \cdots$$

so

$$C(u_0) = \frac{1}{2} + \frac{1}{\pi u_0} \sin \frac{\pi}{2} u_0^2 + \cdots$$

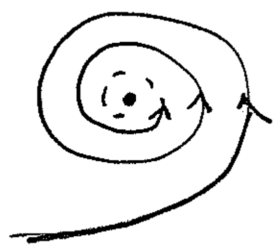
$$S(u_0) = \frac{1}{2} - \frac{1}{\pi u_0} \cos \frac{\pi}{2} u_0^2 + \cdots$$

the curve $C(u_0) + i S(u_0)$ then

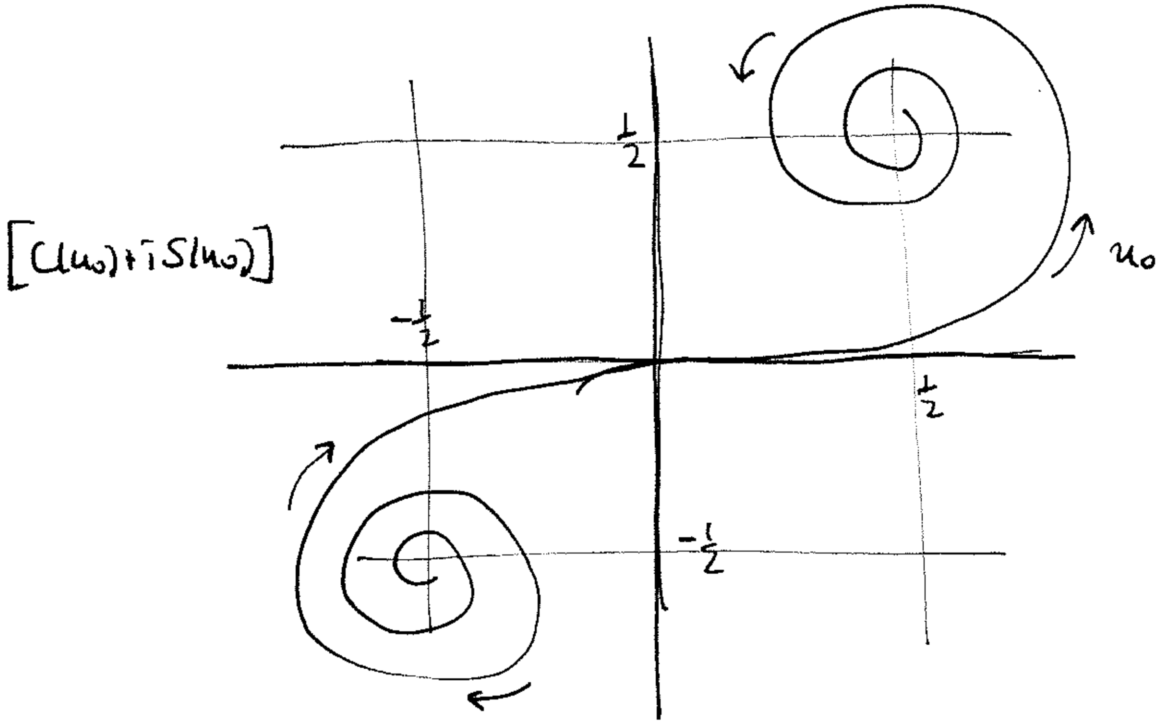
→ passes ~~thru~~ $0+i0$, is symmetric under $z \to -z$, ends up at $\frac{1+i}{2}$ as $u_0 \to \infty$

→ near 0, is flat, like a cubic 

→ near ∞, goes around in the positive direction and slowly falls into $\frac{1}{2} + i\frac{1}{2}$



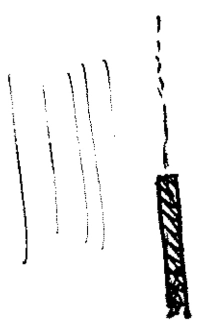
putting the pieces together:



This curve is called the <u>Cornu spiral</u>. (For a more accurate graph, see Fig. 13-2 of Heald & Marion.)

According to p. 4, the intensity of the diffraction pattern is given by the square of the distance between the points $u_0 = u_1$ and $u_0 = u_2$ on the Cornu spiral. Let's investigate this in some illustrative examples.

First, consider a sharp edge



ie $a_1 = 0$ $a_2 \to \infty$

then

$$u_2 \longrightarrow \infty \qquad u_1 = \sqrt{\frac{2}{\lambda z}}\,(-Y)$$

so

$$C(u_2) + i\,S(u_2) = \tfrac{1}{2} + i\,\tfrac{1}{2}.$$

as $\underline{Y \to}\ -\infty$

$$C(u_1) + i\,S(u_1) = \tfrac{1}{2} + i\,\tfrac{1}{2} - i\,\frac{1}{\pi}\sqrt{\frac{\lambda z}{2}}\,\frac{1}{|Y|}\,e^{i\frac{\pi}{2}\frac{2}{\lambda z}Y^2} + \cdots$$

so

$$\left|\big(C(u_2)+i\,S(u_2)\big) - \big(C(u_1)+i\,S(u_1)\big)\right|^2$$

$$\cong \frac{\lambda z}{2\pi^2 Y^2}$$

smooth and $\longrightarrow 0$ as $Y \to -\infty$

at $Y = 0$ $\qquad u_1 = 0$

$$\left|\big(C(u_2)+i\,S(u_2)\big) - \big(C(u_1)+i\,S(u_1)\big)\right|^2 \cong \left|\tfrac{1}{2}+i\tfrac{1}{2}\right|^2 = \frac{1}{2}$$

as $Y \to \infty$ $\qquad u_1 = -\sqrt{\frac{2}{\lambda z}}\,Y \to -\infty$

so

$$\big(C(u_2)+i\,S(u_2)\big) - \big(C(u_1)+i\,S(u_1)\big)$$

$$= 1 + i - \frac{i}{\pi}\sqrt{\frac{\lambda z}{2}}\,\frac{1}{Y}\,e^{i\frac{\pi}{2}\frac{2}{\lambda z}Y^2}$$

to first approximation, the intensity is

$$|1+i|^2 = 2$$

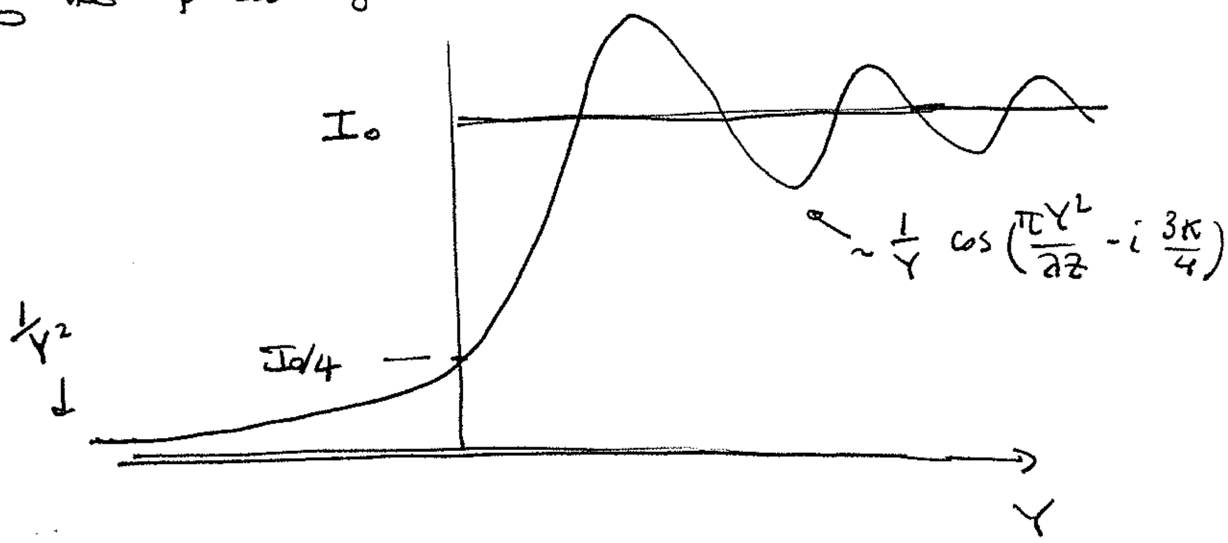
In the next approximation, we have an oscillation about this value:

$$\left| 1+i - \frac{i}{\pi}\sqrt{\frac{\lambda Z}{2}}\frac{1}{Y} e^{i\frac{\pi Y^2}{\lambda Z}} \right|^2$$

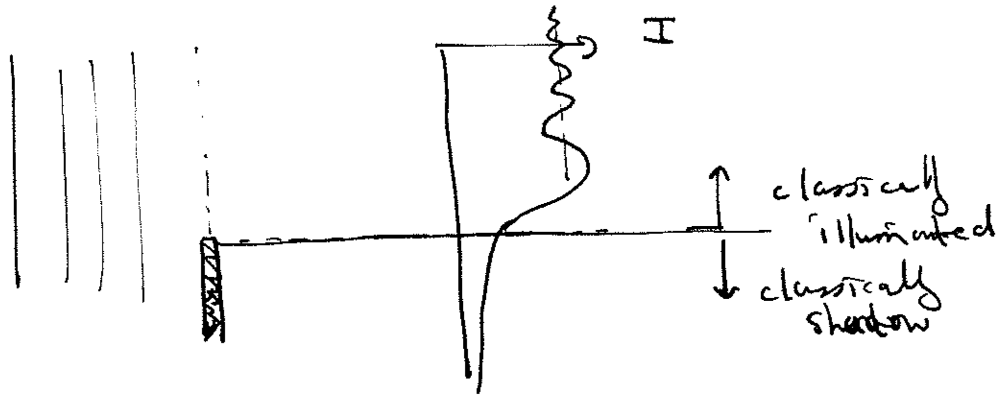
$$= |1+i|^2 \left| 1 - \frac{i}{2\pi}\sqrt{\frac{\lambda Z}{Y^2}} e^{i\frac{\pi Y^2}{\lambda Z} - i\pi/4} \right|^2$$

$$= 2\left(1 + \frac{1}{\pi}\sqrt{\frac{\lambda Z}{Y^2}} \cos\left(\frac{\pi Y^2}{\lambda Z} - i\frac{3\pi}{4}\right) + \cdots \right)$$

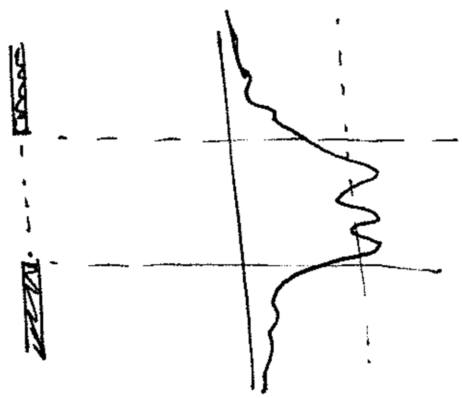
putting the pieces together, the intensity as a function of Y is



This picture nicely generalizes the ray-optics notion of illuminated and shadow regions:



A pattern with a finite slit, in the Fresnel region, looks like



Now let's explore what happens when we send

$$Y, Z \to \infty \qquad Y/Z = \theta \quad \text{fixed.}$$

and go back into the Fraunhofer region. $Y \gg a$ means we are in the geometrical shadow. I'll now set $a_1 = a_2 = a$, to give a symmetrical slit of size $2a$.

$$u_1 = \sqrt{\frac{2Z}{\lambda}}\left(-\theta - \frac{a}{Z}\right) \qquad u_2 = \sqrt{\frac{2Z}{\lambda}}\left(-\theta + \frac{a}{Z}\right)$$

so both $u_1, u_2 \to -\infty$ as $Z \to \infty$ for $\theta > 0$.

$$\Delta u = u_2 - u_1 = \sqrt{\frac{2}{z\lambda}}\ 2a \ll 1$$

if $z \gg a^2/\lambda$

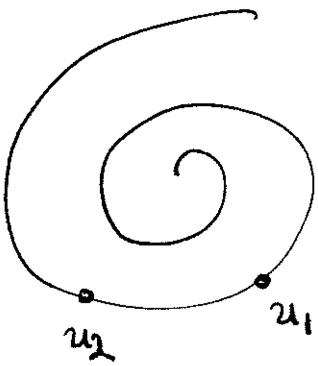
but the argument of the cos is

$$\frac{\pi}{2} u^2$$

and

$$\Delta\left(\frac{\pi}{2}u^2\right) = \pi \cdot \sqrt{\frac{2z}{\lambda}}(-\theta) \cdot \Delta u = \frac{2\pi}{\lambda} \cdot 2a(-\theta)$$

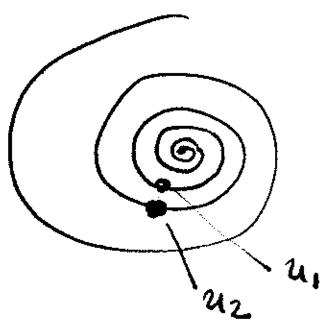
so in the Fraunhofer limit, the two arguments u_2 and u_1 run around the spiral



in such a way that, at the Fraunhofer condition for a minimum

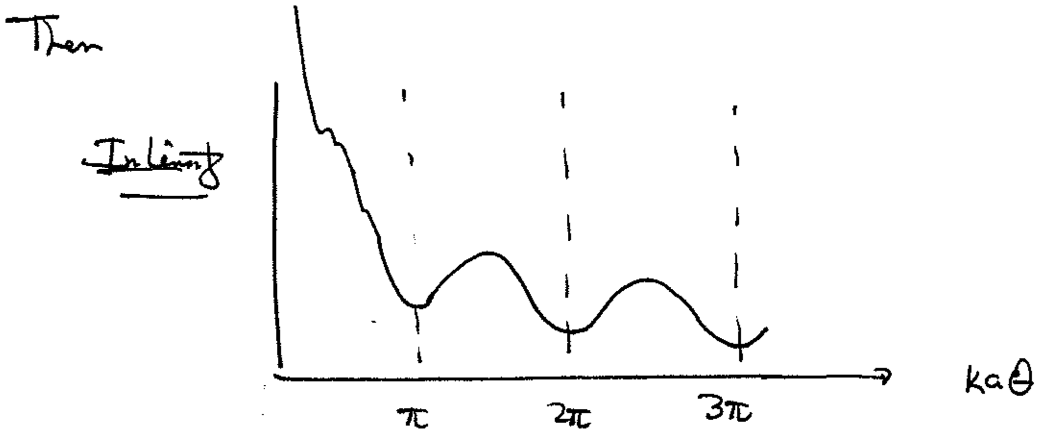
$$ka\theta = n\pi \qquad \text{or} \qquad \frac{2a\theta}{\lambda} = n$$

they are at the same phase angle:



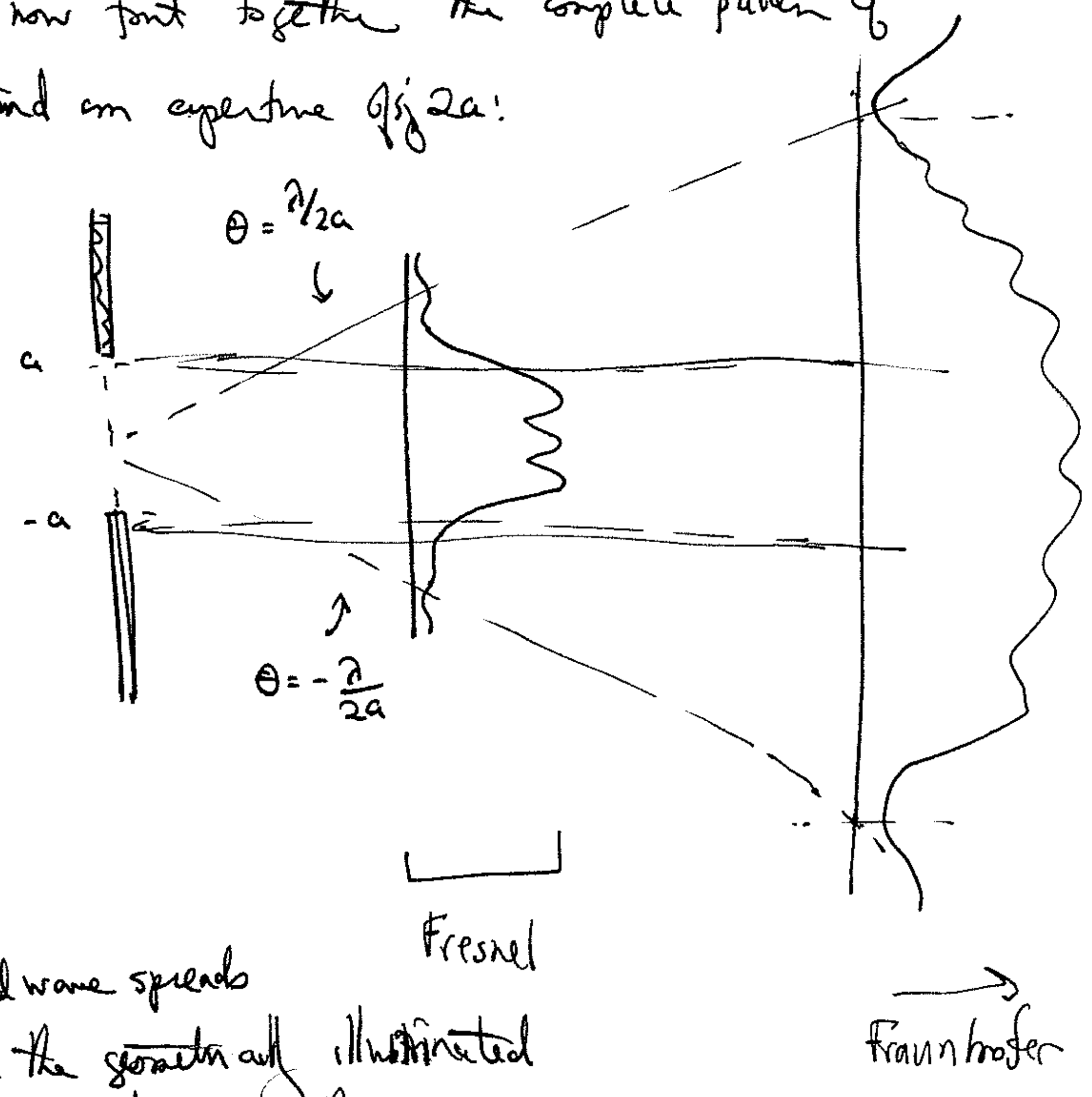
$n = 1$

Then



with the dips coming closer to zero as $z \to \infty$.

We can now put together the complete pattern of the fields behind an aperture of size $2a$:



The diffracted wave spreads out from the geometrically illuminated region to fill in the angular region required for the Fraunhofer diffraction pattern.