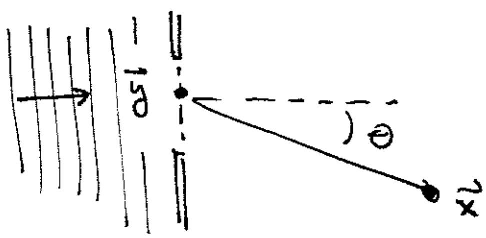
May 14

Fraunhofer Diffraction

In the previous lecture, we studied the diffracted light from a disk in a very symmetrical situation – just on the axis. What happens off the axis ~~a~~ for a more general shape of aperture? To investigate this, we need to make some approximations to the general diffraction integral.

So, go back to:

$$\psi_0(\vec{x}) = \frac{-ik\,\psi_0}{2\pi} \int_{\text{hole}} d^2y \; \frac{e^{ik|x-y|}}{|x-y|} \left(\frac{1+\cos\theta}{2}\right)$$



Consider this integral for $\vec{x}$ well in front of the aperture: If the aperture has size a, $|\vec{x}| \gg a$. This allows us to approximate:

$$\frac{1}{|\vec{x}-\vec{y}|} \to \frac{1}{x} \qquad \left(\frac{1+\cos\theta}{2}\right) \to 1$$

and

$$k|x-y| = k\left(x^2 - 2\vec{x}\cdot\vec{y} + y^2\right)^{1/2} = kx - k\,\frac{\vec{x}\cdot\vec{y}}{x} + \frac{ky^2}{2x} - \frac{1}{8}k\frac{(\vec{x}\cdot\vec{y})^2}{x^3} + \cdots$$

or

$$k|\vec{x}-\vec{y}| = kx - k\,\hat{x}\cdot\vec{y} + \frac{k}{2x}\left[y^2 - (\hat{x}\cdot\vec{y})^2\right] + \cdots$$

For sufficiently large x, we can ignore the third term here. The condition for this is

$$\frac{k}{x}y^2 \ll 1 \quad \text{or} \quad \frac{a}{\lambda}\cdot\frac{a}{|\vec{x}|} \ll 1$$

i.e. $|\vec{x}| \gg \dfrac{a^2}{\lambda}$, the Fraunhofer limit.

Later we will see what happens when we come closer in and keep the third term above; that is the regime of Fresnel diffraction.

For a wave received at $\vec{x}$, the outgoing wavevector is

$$\vec{k} = k\hat{x}$$

In the Fraunhofer limit, the radiation field is then

$$\psi_0(\vec{x}) = \frac{-ik\psi_0}{2\pi}\,\frac{e^{ikx}}{x}\int_{\text{hole}} d^2y\; e^{-i\vec{k}\cdot\vec{y}}$$

Thus, the radiation amplitude is just proportional to the Fourier transform of the aperture! For complementary screens 1 and 2

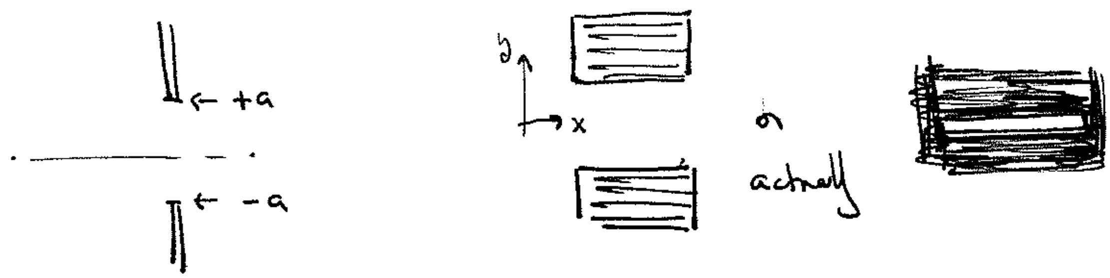
$$\int_1 d^2y\, e^{-i\vec{k}\cdot\vec{y}} + \int_2 d^2y\, e^{-i\vec{k}\cdot\vec{y}}$$

$$= \int_{\substack{\text{whole}\\ \text{plane}}} d^2y\, e^{-i\vec{k}\cdot\vec{y}} = (2\pi)^2\,\delta(k_x)\,\delta(k_y)$$

so away from the forward direction $\psi_0(\vec{x})\big|_1 = -\psi_0(\vec{x})\big|_2$

as required by Babinet's principle.

Let's now evaluate the Fraunhofer diffraction formula for some simple shapes. For a long slit of width 2a



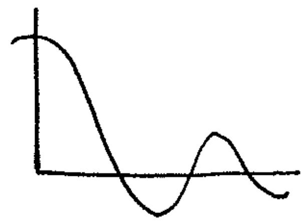
the eventual pattern is narrow in $\hat{x}$ but, in $\hat{y}$, has the form

$$\int_{-a}^{a} dy\, e^{-ik_y y} = \frac{1}{-ik_y}\left(e^{-ik_y a} - e^{+ik_y a}\right)$$

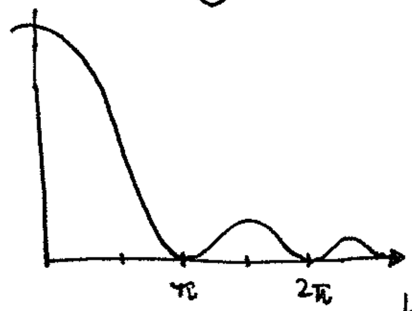
$$= 2\left(\frac{\sin k_y a}{k a}\right)$$

Then, as a function of $k_y = k\sin\Theta$ at $x=0$

$$\psi_0 \sim \frac{\sin k_y a}{k_y a}$$



~~Intensity~~ $\sim \left(\frac{\sin k_y a}{k_y a}\right)^2$



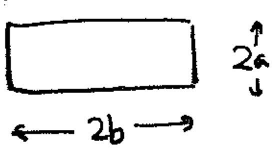
90% of the area under this curve lies in the central lobe.

The angular spread of the diffracted beam

$$\Delta \sin\Theta \text{ or } \Delta\Theta \sim \frac{\pi}{ka} \sim \frac{\lambda}{2a}$$

This angular spread becomes large when $a \sim \lambda$

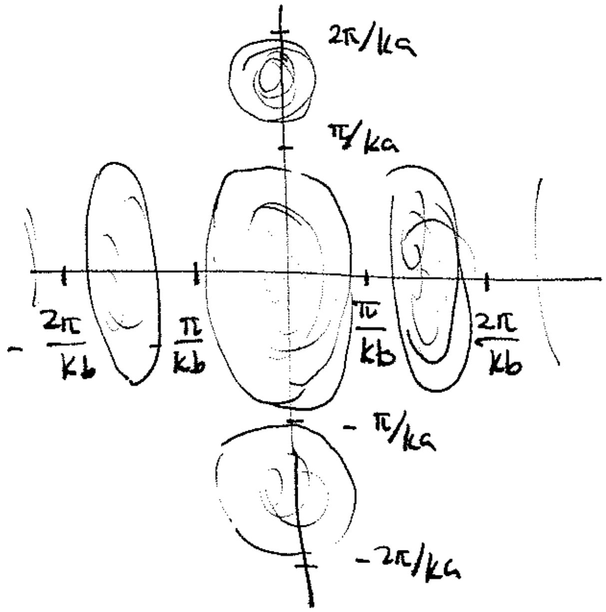
For a two-dimensional aperture:



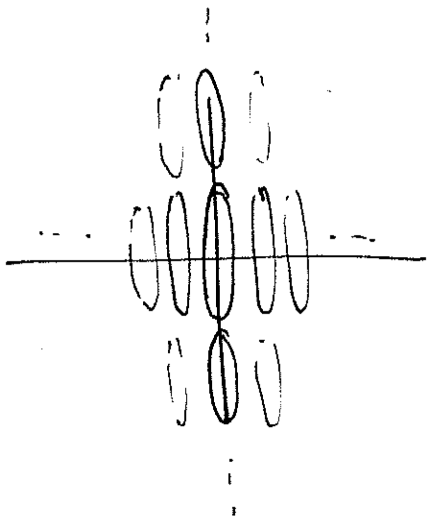
we evaluate this integral for each dimension:

$$I \sim \left(\frac{\sin k_x b}{k_x b}\right)^2 \left(\frac{\sin k_y a}{k_y a}\right)^2$$

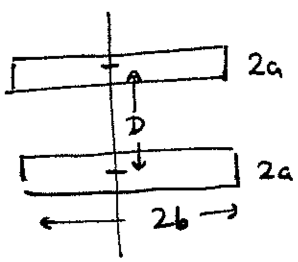
Projected onto a screen some distance away



again



A pattern of two parallel slits:



gives a wave amplitude proportional to

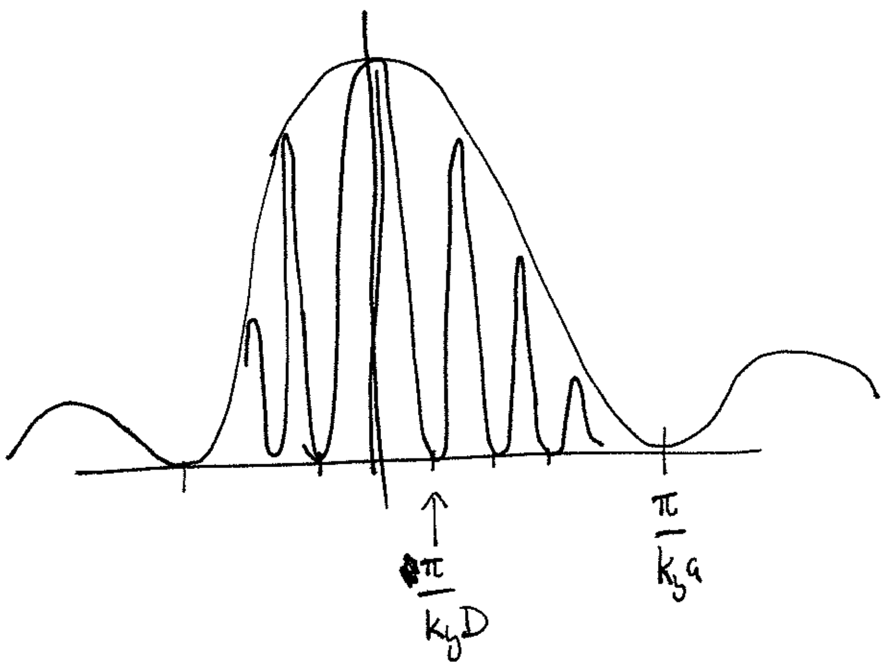
$$\int\limits_{\substack{-b<x<b \\ -a<y<a}} d^2y\, e^{-i\vec{k}\cdot\vec{y}} + \int\limits_{\substack{-b<x<b \\ -a<y<a}} d^2y\, e^{-i\vec{k}\cdot\vec{y}}\, e^{-i\vec{k}\cdot D\hat{y}}$$

$$= \int\limits_{\text{single slit}} d^2y\, e^{-i\vec{k}\cdot\vec{y}} \left(1 + e^{-ik_y D}\right)$$

$$\propto \left(\frac{\sin k_x b}{k_x b}\right)\left(\frac{\sin k_y a}{k_y a}\right)\left(2 \cos k_y D/2\right)$$

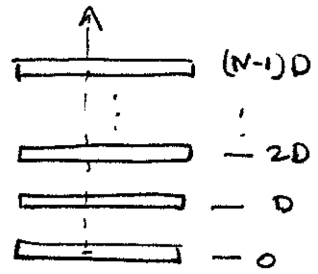
the intensity pattern is the square of this. The intensity pattern in y is then modulated by

$$4 \sin^2(k_y D/2)$$

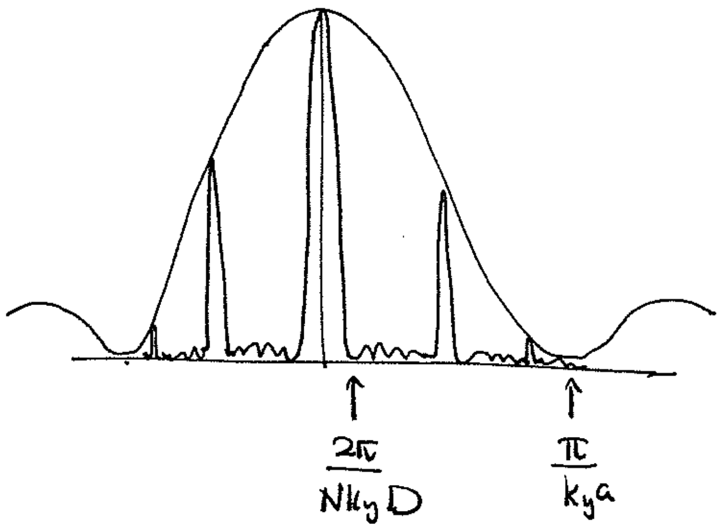


The modulating factor $4\cos^2 k_y D/2 = \left|\frac{\sin 2k_y D/2}{\sin k_y D/2}\right|^2$, the 2-slit diffraction pattern

For N slits:



$$I \sim \left(\frac{\sin k_x b}{k_x b}\right)^2 \left(\frac{\sin k_y a}{k_y a}\right)^2 \cdot \left(\frac{\sin N k_y D/2}{\sin k_y D/2}\right)^2$$



The diffraction pattern of the slit supplies an overall modulation to the N-source diffraction pattern.

Now let's consider the case of a circular aperture

$$\psi_0(\vec{x}) = \frac{-ik\psi_0}{2\pi} \frac{e^{ikx}}{x} \int_{\text{hole}} d^2y \; e^{-i\vec{k}\cdot\vec{y}}$$

The diffraction pattern is cylindrically symmetric. Write the integral for $\vec{k}$ in the $\hat{x}\hat{z}$ plane

$$\int_0^a dr\, r \int_0^{2\pi} d\phi \quad e^{-i k \sin\theta \cos\phi \cdot r}$$

$$= \int dr\, r \cdot 2\pi \cdot \int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{-i(kr\sin\theta)\cos\phi}$$

$$= 2\pi \int_0^a dr\, r \quad J_0(kr\sin\theta)$$

To integrate this, you might want to know that

$$\frac{1}{z}\frac{d}{dz}\left(z^n J_n(z)\right) = z^{n-1} J_{n-1}(z)$$

i.e.

$$\frac{d}{dz}\left(z\, J_1(z)\right) = z\, J_0(z)$$

so the above is:

$$= \frac{2\pi}{(k\sin\theta)^2}\,(ka\sin\theta)\; J_1(ka\sin\theta)$$

so

$$\psi_0(\vec{x}) = -i\,\psi_0\,\frac{e^{ikx}}{x}\cdot\frac{ka^2}{2}\cdot\left(\frac{2J_1(u)}{u}\right)$$

where $\quad u = (ka\sin\theta)$

the intensity pattern is

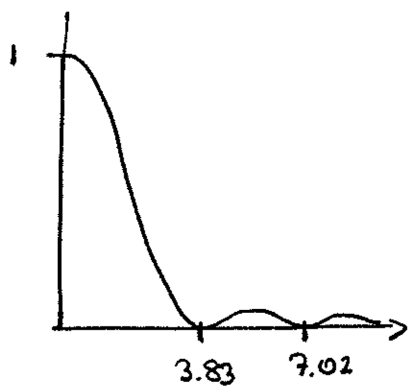
$$I \propto \left(\frac{2 J_1(u)}{u} \right)^2$$

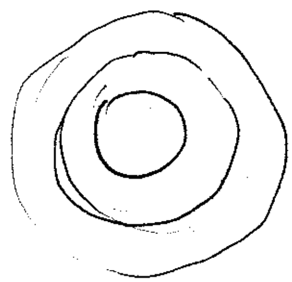
now

$$J_1(u) \sim \frac{u}{2} \quad \text{as} \quad u \to 0$$

$$\sim \sqrt{\frac{2}{\pi u}} \cos\left(u - \frac{3\pi}{4}\right) \quad u \to \infty$$

so we obtain a pattern: "Airy diffraction pattern"





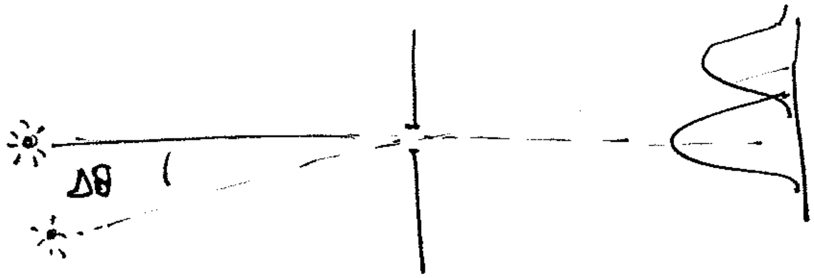
$u = ka \cdot \theta$

The central circle contains 84% of the total power.

For small angles, the first minimum is at

$$\theta = \frac{3.83}{ka} = 0.61 \frac{\lambda}{a}$$

Now think about a telescope or camera with a circular aperture looking at two distant point sources separated by an angle $\Delta\theta$



each source creates a spot behind the circular aperture of angular radius $0.61 \frac{\lambda}{a}$. The sources can be resolved (according to "Rayleigh's criterion") if the center of one source is beyond the first zero from the other

$$\Delta\theta > 0.61 \frac{\lambda}{a}$$

This criterion is also called the "diffraction limit" of resolution.

To put some numbers into this, consider a wavelength in the visible spectrum: 5000 Å (green) = 5×10^{-7} m. Then with a 1mm (or 10cm) aperture, we can resolve

$$1\text{mm} \rightarrow \Delta\theta = 3 \times 10^{-4} \qquad 10\text{cm} \rightarrow \Delta\theta = 3 \times 10^{-6}$$

This corresponds to

1mm aperture → 0.3m at 1 km
10 cm aperture → 0.3m at 100 km

so if you would like to spy from space, it is possible with a sufficiently large camera.